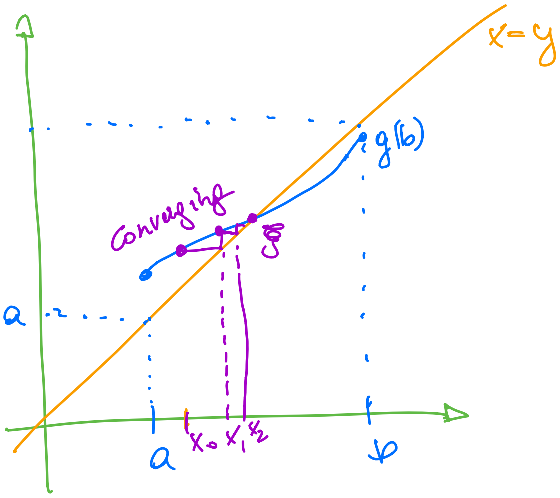
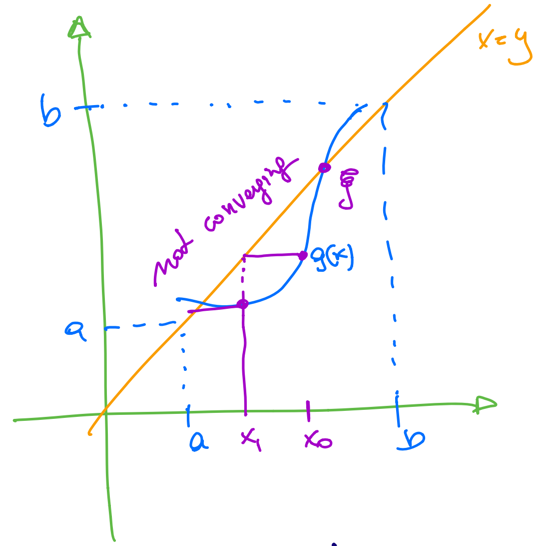




Thm: Consider fixed point iteration (*) for g, g contraction on $[a,b]$, fixed point ξ. Then, for $\varepsilon > 0$ we have $|x_k - \xi| \leq \varepsilon$ for all $k \geq k_0(\varepsilon)$ for a $k_0(\varepsilon)$ satisfying

$$k_0(\varepsilon) \leq \left[\frac{\ln|x_1 - x_0| - \ln\left(\varepsilon(1-L)\right)}{\ln\left(1/L\right)} \right] + 1$$

$[x]$ is largest integer $\leq x$.

Proof: (Sketch)

$$|x_0 - \xi| = |x_0 - x_1 + x_1 - \xi| \leq |x_0 - x_1| + |x_1 - \xi|$$
$$\leq |x_0 - x_1| + L|x_0 - \xi|$$

$$\Longrightarrow |x_0 - \xi| \leq \frac{1}{1-L}|x_0 - x_1|$$

$$\Longrightarrow |x_k - \xi| \leq L^k|x_0 - \xi| \leq \underbrace{\frac{L^k}{1-L}|x_0 - x_1|}$$

make $\leq \varepsilon$, then $|x_k - \xi| \leq \varepsilon$ by choosing k large enough (take logarithm)

$\square$.

Computing Lipschitz constant L:

$$g : [a,b] \to \mathbb{R}, \quad g(x) \in [a,b] \text{ f. all } x \in [a,b]$$

$$|g(x) - g(y)| \leq L\,|x-y|$$

Mean value theorem (assuming g is differentiable):

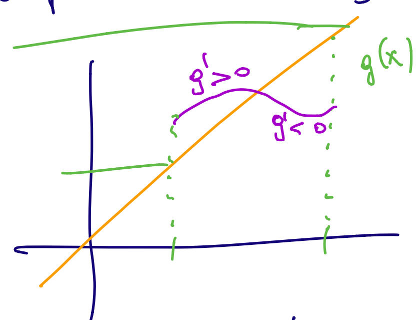
$$\frac{g(x)-g(y)}{x-y} = g'(\eta) \qquad \eta \in (x,y) \subset [a,b]$$

$$\Longrightarrow \frac{|g(x)-g(y)|}{|x-y|} = |g'(\eta)| \qquad \text{— '' —}$$

$$\boxed{L := \max_{\eta \in [a,b]} |g'(\eta)|} \Longrightarrow |g(x)-g(y)| \leq L\,|x-y|$$

method to compute L on $[a,b]$.

Example: Consider $f(x) = 0$ on $[1,2]$
with $f(x) = e^x - 2x - 1$
Equivalent fixed point equation
$$g(x) = \ln(2x+1)$$



g is diff'able $\quad g'(x) = \frac{2}{2x+1}, \quad g''(x) = -\frac{4}{(2x+1)^2} < 0$ on $[1,2]$

$$g'(1) \geq g'(\eta) \geq g'(2) \quad \text{for } \eta \in [1,2]$$
(because $g' \downarrow$)

$$\Longrightarrow \frac{2}{3} \geq g'(\eta) \geq \frac{2}{5}$$

$$\Longrightarrow |g'(\eta)| \leq \frac{2}{3} \text{ for all } \eta \in [1,2]$$

$$\Longrightarrow |g(x)-g(y)| \leq L\,|x-y| \text{ f. all } x,y \in [1,2], \ L := \frac{2}{3}$$

$\Longrightarrow$ convergence from any $x_0 \in [1,2]$ to unique fixed point.

There is also a local version of the convergence result:

Thm: Assumptions as before and g is continuously differentiable, ξ fixed point with $|g'(\xi)| < 1$.
Then the fixed point iteration converges to ξ provided x_0 is sufficiently close to ξ.

Proof: (Sketch) If $|g'(\xi)| < 1 \Longrightarrow$ there exists $h > 0$ such that $|g'(x)| < 1$ for all $x \in [\xi - h, \xi + h]$; g is a contraction in this interval and if x_0 is chosen in it, the algorithm converges following the previous theorem. $\square$.