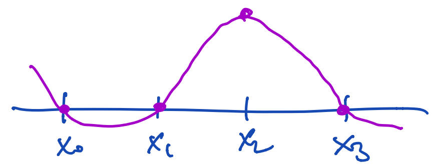
Lagrange interpolation:

Lagrange polynomials for nodes $x_0, x_1, \dots x_n$ $(x_i \neq x_j,\ i \neq j)$

$L_k(x) \in P_n$ ... space of polynomials of degree $\leq n$

$$L_k(x_i) = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$



$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k \in P_n$$ .. interpolating polynomial for $(x_0, y_0), \dots (x_n, y_n)$.

Lemma: $n \geq 1$, For given distinct points $x_0, \dots x_n$ and $y_0, \dots, y_n$ there exists a unique $p_n \in P_n$ such that $p_n(x_i) = y_i$ for $i = 0, \dots, n$.

Proof: Existence ✓

Unique: Let $p_n, q_n \in P_n$ be interpolating polynomials

$\longrightarrow p_n(x_i) - q_n(x_i) = y_i - y_i = 0 \qquad i = 0, \dots, n$

$p_n - q_n \in P_n$ with $(n+1)$ roots

$\longrightarrow p_n - q_n = 0 \Longrightarrow p_n = q_n.$ $\square$

The $\boxed{p_n = \sum_{k=0}^{n} L_k(x) y_k}$ is the unique Lagrange interpolation polynomial

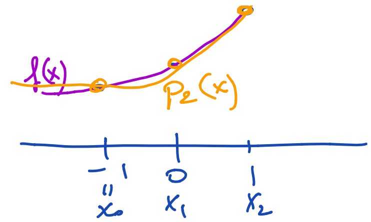
Given $f: \mathbb{R} \to \mathbb{R}$, then

$\boxed{p_n = \sum_{k=0}^{n} L_k(x) f(x_k)}$ is the unique Lagrange polynomial that interpolates f.

Example: $f: x \longmapsto e^x$ on $[-1, 1]$, $x_0 = -1,\ x_1 = 0,\ x_2 = 1$

$$L_0(x) = \prod_{\substack{k=0\\k\neq 0}}^{2} \frac{(x-x_k)}{(x_0-x_k)} = \frac{(x-0)(x-1)}{(-1-0)(-1-1)}$$

$$= \frac{1}{2}x(x-1)$$



$$L_1(x) = 1-x^2, \quad L_2(x) = \frac{1}{2}x(x+1)$$

$$P_2(x) = \frac{1}{2}x(x-1)e^{-1} + (1-x^2)\cdot 1 + \frac{1}{2}x(x+1)e^{1}$$

$$= 1 + x\sinh 1 + x^2(\cosh 1 - 1) \in \mathcal{P}_2$$

Theorem: $n \geq 1$, $f:[a,b] \to \mathbb{R}$, $(n+1)$st derivative of f exists and is continuous. Then for $x \in [a,b]$ exists $\xi \in (a,b)$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\pi_{n+1}(x)$$

$$\pi_{n+1}(x) = (x-x_0)\cdot \ldots \cdot (x-x_n)$$

and:

$$|f(x)-p_n(x)| \leq \frac{M_{n+1}}{(n+1)!}\,|\pi_{n+1}(x)|,$$

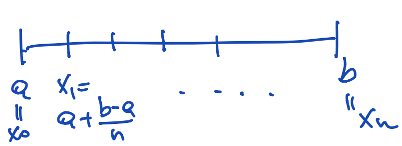
$$M_{n+1} = \max_{x\in[a,b]} |f^{(n+1)}(x)|$$

§6.3 Convergence

Does $p_n(x)$ converge to f as $n \to \infty$, and in what sense?

Answer: Not always, in particular it depends on the choice of the nodes $x_0, x_1 \ldots x_n$

Assume equally spaced points $x_j = a + j\frac{(b-a)}{n}$ $j = 0,1,\ldots,n$



What happens to $\frac{M_{n+1}}{(n+1)!} \max_{x\in[a,b]} |\pi_{n+1}(x)|$?

It can happen that this does not go to zero as $M_{n+1} \max_{x\in[a,b]} |\pi_{n+1}(x)|$ goes to ∞ faster than $(n+1)!$.

§ 6.4 Hermite interpolation

$x_0, \dots, x_n \in \mathbb{R}$, $x_i \neq x_j$ for $i \neq j$

$y_0, \dots, y_n \in \mathbb{R}$ Find $P_{2n+1} \in \mathcal{P}_{2n+1}$ such that

$z_0, \dots, z_n \in \mathbb{R}$

$$P_{2n+1}(x_i) = y_i \quad i = 0, \dots, n$$

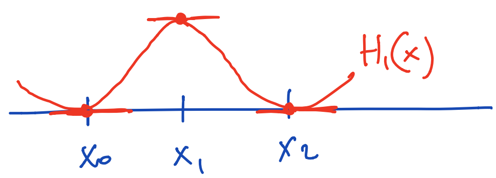
Theorem: (Hermite interpolation)

$$P'_{2n+1}(x_i) = z_i \quad i = 0, \dots, n$$

Consider $$H_k(x) = (L_k(x))^2 \left(1 - 2L'_k(x_k)(x - x_k)\right) \in \mathcal{P}_{2n+1}$$

$$K_k(x) = L_k(x)^2 (x - x_k) \qquad \in \mathcal{P}_{2n+1}$$

H_k, K_k satisfy: $$H_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H'_k(x_i) = 0$$



$K_k(x_i) = 0$, $$K'_k(x_i) = \begin{cases} 1 & i = k \\ 0 & \text{else} \end{cases}$$

Using $H_k(x)$, $K_k(x)$, the Hermite interpolation is

$$P_{2n+1}(x) = \sum_{k=0}^{n} \left(H_k(x)\, y_k + K_k(x)\, z_k \right)$$