Example: $x = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \in \mathbb{R}^4 \longrightarrow$ transform to multiple of e_1

$$v = x + \operatorname{sgn}(1)\sqrt{10}\, e_1 = \begin{pmatrix} 1+\sqrt{10} \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \frac{2}{\|v\|^2} \begin{pmatrix} 1+\sqrt{10} \\ 2 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1+\sqrt{10} & 2 & 1 & 2 \end{pmatrix}$$

We can also use Householder in a subspace given by e_2, e_3, e_4, i.e. we want to find H such that

$$H \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \pm * \\ 0 \\ 0 \end{pmatrix} \qquad = \left\| \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\| = 3$$

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \operatorname{sign}(2) \cdot 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

$$v = \begin{pmatrix} 0 \\ 5 \\ 1 \\ 2 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \frac{2}{30} \begin{pmatrix} 0 \\ 5 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 0 & 5 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & -\frac{1}{3} & \frac{14}{15} & -\frac{2}{15} \\ 0 & -\frac{2}{3} & -\frac{2}{15} & \frac{11}{15} \end{pmatrix}$$

Theorem: Given $A \in \mathbb{R}^{n \times n}$, $A^T = A$, $n \geq 3$, Then there exists $Q_n \in \mathbb{R}^{n \times n}$, a product of $n-2$ Householder matrices

$$Q_n = H_{(n,n-1)} \cdot H_{(n,n-2)} \cdot \ldots H_{(n,2)}$$

such that $Q_n^T A Q_n = T_n$ is tridiagonal.

Proof:

$$A = \begin{bmatrix} \alpha & \text{---} b^T \text{---} \\ | & \\ b & C \\ | & \end{bmatrix} \xrightarrow[\text{from left.}]{H_{(n,n-1)}} \begin{bmatrix} \alpha & \text{---} b^T \text{---} \\ * & \\ 0 & C' \\ | & \\ 0 & \end{bmatrix} \longrightarrow$$

$$\xrightarrow[\text{from the right}]{H_{(n,n-1)}} \begin{bmatrix} \alpha & * & 0 & \cdots & 0 \\ * & & & & \\ 0 & & C'' & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix} \xrightarrow[\text{from left and right}]{H_{(n,n-2)}} \begin{bmatrix} \alpha & * & 0 & \cdots & 0 \\ * & * & * & 0 \cdots & 0 \\ 0 & * & & & \\ \vdots & 0 & & D'' & \\ 0 & 0 & & & \end{bmatrix}$$

$$HA = \begin{pmatrix} \alpha & — & b^T & — \\ * & & & \\ 0 & & * & \\ \vdots & & & \\ 0 & & & \end{pmatrix}$$

$(HA)^T = AH$ so multiplication from the right does the same to the rows as multiplication from left to columns.

$$\xrightarrow{\text{repeat}} \begin{bmatrix} * & * & & & \\ * & * & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & & \alpha \\ 0 & & & \alpha & * \end{bmatrix}$$

$$\longrightarrow \underbrace{H_{(n,2)} \cdots H_{(n,n-2)} H_{(n,n-1)}}_{Q_n^T} A \underbrace{H_{(n,n-1)} H_{(n,n-2)} \cdots H_{(n,2)}}_{Q_n} \quad \checkmark$$

How many flops? $\frac{1}{3} n^3$ flops for tridiagonalization