

Example:  $x = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \in \mathbb{R}^4 \rightarrow$  transform to multiple of  $e_1$

$$v = x + \text{sgn}(1) \sqrt{10} e_1 = \begin{pmatrix} 1 + \sqrt{10} \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \frac{2}{\|v\|^2} \begin{pmatrix} 1 + \sqrt{10} \\ 2 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 + \sqrt{10} & 2 & 1 & 2 \end{pmatrix}$$

We can also use Householder in a subspace given by  $e_2, e_3, e_4$ , i.e. we want to find  $H$  such that

$$H \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ * \\ 0 \\ 0 \end{pmatrix} \leftarrow = \left\| \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\| = 3$$

$$w_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \text{sign}(2) \cdot 3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

$$v = \begin{pmatrix} 0 \\ 5 \\ 1 \\ 2 \end{pmatrix} \quad H = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} - \frac{2}{30} \begin{pmatrix} 0 \\ 5 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 0 & 5 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & -\frac{1}{3} & \frac{4}{5} & \frac{2}{5} \\ 0 & -\frac{2}{3} & -\frac{2}{5} & \frac{11}{15} \end{pmatrix}$$

Theorem: Given  $A \in \mathbb{R}^{n \times n}$ ,  $A^T = A$ ,  $n \geq 3$ , Then there exists  $Q_n \in \mathbb{R}^{n \times n}$ , a product of  $n-2$  Householder matrices

$$Q_n = H_{(n,n-1)} \cdot H_{(n,n-2)} \cdot \dots \cdot H_{(n,2)}$$

such that  $Q_n^T A Q_n = T_n$  is tridiagonal.

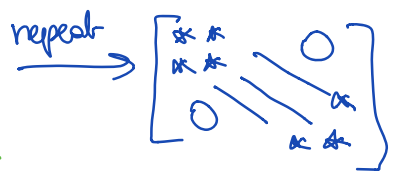
Proof:

$$A = \begin{bmatrix} \alpha & -b^T \\ b & C \end{bmatrix} \xrightarrow[\text{from left.}]{H_{(n,n-1)}} \begin{bmatrix} \alpha & -b^T \\ 0 & C' \end{bmatrix} \rightarrow$$

$$\begin{array}{c}
 H_{(n,n-1)} \\
 \xrightarrow{\text{from the right}} \\
 \left[ \begin{array}{cccc}
 \alpha & * & 0 & \dots & 0 \\
 * & & & & \\
 0 & & C'' & & \\
 \vdots & & & & \\
 0 & & & & 
 \end{array} \right]
 \end{array}
 \xrightarrow{\text{from left and right}}
 \begin{array}{c}
 H_{(n,n-2)} \\
 \left[ \begin{array}{cccccc}
 \alpha & * & 0 & \dots & 0 \\
 * & * & * & 0 & \dots & 0 \\
 0 & * & & D'' & & \\
 \vdots & 0 & & & & \\
 0 & 0 & & & & 
 \end{array} \right]
 \end{array}$$

$$HA = \begin{pmatrix} \alpha & -b^T \\ * & * \\ \vdots & \\ 0 & \end{pmatrix}$$

$(HA)^T = AH$  so multiplication from the right does the same to the rows as multiplication from left to columns.



$$\xrightarrow{\quad} \underbrace{H_{(n,2)} \dots H_{(n,2)} H_{(n,2)} H_{(n,1)}}_{Q_n^T} A \underbrace{H_{(n,n-1)} H_{(n,n-2)} \dots H_{(n,2)}}_{Q_n} \quad \checkmark$$

How many flops?  $\frac{1}{3} n^3$  flops for tridiagonalization