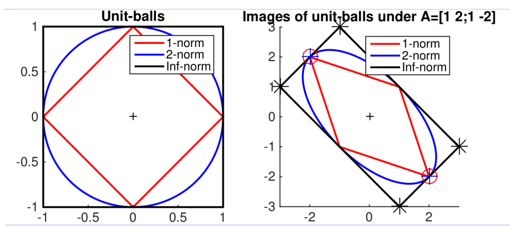
Matrix norm induced by vector norms:

$\|\cdot\|$ norm on $\mathbb{R}^n$ (e.g. $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$)

induces a matrix norm for $A \in \mathbb{R}^{n \times n}$:

$$\|A\| = \max_{\substack{v \in \mathbb{R}^n \\ v \neq 0}} \frac{\|Av\|}{\|v\|} = \max_{\|v\|=1} \|Av\|$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$$



<u>Thm:</u> Induced matrix norm for $\|\cdot\|_\infty$ is the "largest row sum":

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

<u>Proof:</u> $v \in \mathbb{R}^n$, $v \neq 0$

$$|(Av)_i| = \left| \sum_{j=1}^{n} a_{ij} v_j \right| \le \sum_{j=1}^{n} |a_{ij}| |v_j| \le$$

$$\le \|v\|_\infty \sum_{j=1}^{n} |a_{ij}|$$

$$\rightarrow \frac{\|Av\|_\infty}{\|v\|_\infty} = \frac{\max_i |(Av)_i|}{\|v\|_\infty} \le \max_i \sum_{j=1}^{n} |a_{ij}| =: C$$

$\Longrightarrow \|A\|_\infty \leq C$

Next, show $\|A\|_\infty \geq C$; we choose $v \in \mathbb{R}^n$ which makes all estimates above "$=$" instead of "$\leq$".

$v_j = \text{sign}(a_{mj})$, where m is the index where maximum in C is obtained, $\|v\|_\infty = 1$

$$\longrightarrow \|Av\|_\infty = \max_i \left|\sum_{j=1}^n a_{ij} v_j\right| \geq \left|\sum_{j=1}^n a_{mj} v_j\right|$$

$$= \sum_{j=1}^n |a_{mj}| = C$$

$$\Longrightarrow \frac{\|Av\|_\infty}{\|v\|_\infty} \geq C \Longrightarrow \boxed{\|A\|_\infty = C}$$

$\square$.

Recall: • The induced norm by the $\|.\|_1$-norm is the largest column sum, i.e.:

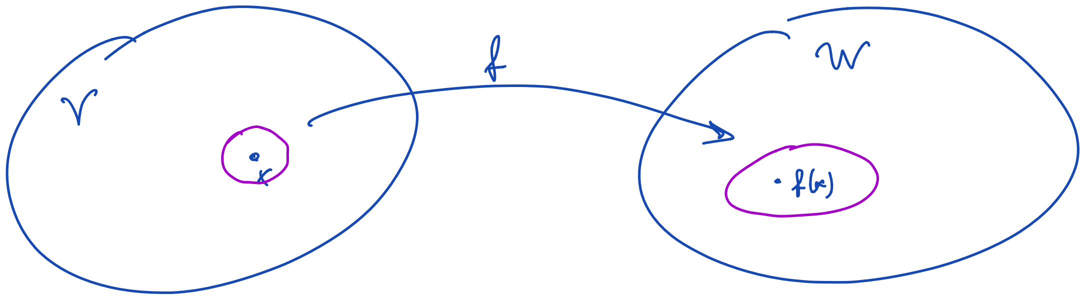
$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$$

• Induced by the $\|.\|_2$-norm is

$$\|A\|_2 = \max_{1 \leq i \leq n} \sqrt{\lambda_i}, \quad \lambda_i \text{ are eigenvalues of } A^T A.$$

Condition numbers: measure of sensitivity of output to changes in the input.

Let $f: (V, \|.\|_V) \longrightarrow (W, \|.\|_W)$ map between two normed vector spaces



Absolute local condition number:

$$\mathrm{cond}_x^{(a)}(f) = \sup_{\substack{\delta x \in V \\ \delta x \neq 0 \\ \delta x \to 0 \\ x+\delta x \in V}} \frac{\| f(x+\delta x) - f(x) \|_W}{\| \delta x \|_V}$$

Relative local condition number:

$[x \neq 0,\ f(x) \neq 0]$

$$\mathrm{cond}_x^{(r)}(f) = \sup_{\substack{\delta x \in V \\ \delta x \neq 0 \\ \delta x \to 0 \\ x+\delta x \in V}} \frac{\| f(x+\delta x) - f(x) \|_W \,/\, \| f(x) \|_W}{\| \delta x \|_V \,/\, \| x \|_V}$$

<u>Remarks:</u>
- depends on the choice of norms
- $\mathrm{cond}_x f \gg 1$ ill-conditioned

  $\mathrm{cond}_x f \sim 1$ well-conditioned

Condition number of

$$f = A^{-1} : \overset{x =}{b} \in \mathbb{R}^n \longrightarrow \overset{y =}{\overbrace{A^{-1} b}} = x \in \mathbb{R}^n$$

norm in $\mathbb{R}^n$, induced matrix norm:

$$\mathrm{cond}_b^{(r)}(A^{-1}) = \sup_{\substack{\delta b \in \mathbb{R}^n \\ \delta b \neq 0 \\ \delta b \to 0}} \frac{\| A^{-1}(b+\delta b) - A^{-1} b \| \,/\, \| A^{-1} b \|}{\| \delta b \| \,/\, \| b \|}$$

$$= \sup_{\delta b \neq 0} \frac{\|A^{-1}\delta b\| / \|A^{-1}b\|}{\|\delta b\| / \|b\|} = \|A^{-1}\| \frac{\|b\|}{\|A^{-1}b\|} \overset{(*)}{\leq}$$

$$\left[b = A(A^{-1}b) \Longrightarrow \underset{\text{vector norms}}{\|b\| \leq \overset{\text{ind. matrix norm}}{\|A\|} \, \|A^{-1}b\|} \right]$$

$$\overset{(*)}{\leq} \|A^{-1}\| \, \|A\|$$

Def: condition number of a regular matrix A is

$$\kappa(A) = \|A\| \|A^{-1}\|$$

The condition number of A depends on the norm used:

$$\kappa_1(A) = \|A\|_1 \, \|A^{-1}\|_1$$

$$\kappa_2(A) = \|A\|_2 \, \|A^{-1}\|_2$$

$$\vdots$$

Ex: $A = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & & & 1 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ \vdots & & \ddots & \\ -1 & & & 1 \end{bmatrix}$

$\|A\|_1 = n \quad \|A^{-1}\|_1 = n \longrightarrow \kappa_1(A) = n^2$

$\|A\|_\infty = 2 \quad \|A^{-1}\|_\infty = 2 \qquad \kappa_\infty(A) = 4$

A system can be well-conditioned with respect to one norm, but not well-conditioned w.r.to another norm.

Theorem: $A \in \mathbb{R}^{n \times n}$ regular, $b \in \mathbb{R}^n$, $Ax = b$, $A(x + \delta x) = b + \delta b$

$$\Longrightarrow \frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}$$