Summary of Newton's method to solve \( f(x) = 0 \), \( f: \mathbb{R} \Rightarrow \mathbb{R} \)

Choose \( x_0 \in [a,b] \) (initialization)

\[
x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{f(x_k)}{f'(x_k)} f(x_k) \quad k = 0, 1, 2, \ldots
\]

Theorem: (Convergence of Newton's method)\[ \]
\[ f \text{ cont, } f'' \text{ continuous on } I_0 = [\overline{x} - \delta, \overline{x} + \delta], \delta > 0 \]
\[ f(\overline{x}) = 0, \quad f''(\overline{x}) \neq 0, \quad \exists A > 0: \]
\[
\left| \frac{f''(x)}{f'(y)} \right| \leq A \quad \text{for all } x, y \in I_0
\]

If \( |x - \overline{x}| \leq h := \min \left( \delta, \frac{1}{A} \right) \), then the Newton sequence \( x_k \rightarrow \overline{x} \) quadratically.

\[
\frac{|x_{k+1} - \overline{x}|}{|x_k - \overline{x}|^2} \rightarrow \gamma < \infty
\]

Remark: implies that \( f'(\overline{x}) \neq 0 \).

Newton's method requires \( f'(x_k) \), in every iteration.

Example: Newton to solve \( f(x) = x^3 = 0 \)

Obvious solution: \( x = 0 \)

§ 1.5 Secant method

does not require \( f' \), as it approximates derivatives by difference quotients:
\[ f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \quad \text{if } x_k \neq x_{k-1} \]

Secant method:

Initialization: \( x_0, x_1 \)

\[ x_{k+1} = x_k - \frac{f(x_k) - f(x_{k-1})}{f(x_k) - f(x_{k-1})} x_k - x_{k-1}, \quad k = 1, 2, 3, \ldots \]

**Theorem:** If continuously differentiable on \([\bar{a} - h, \bar{a} + h], h \geq 0\)
\[ f(\bar{a}) = 0, \quad f'(\bar{a}) \neq 0 \]

Then the secant method converges at least linearly to \( \bar{a} \) if \( x_0, x_1 \) are sufficiently close to \( \bar{a} \).

**Remark:** One can show that the secant method converges faster:
\[ \lim_{k \to \infty} \frac{|x_{k+1} - \bar{a}|}{(x_{k+1} - \bar{a})^q} = \mu < \infty \]

for \( q_n = \frac{1}{2} \left( 1 + \sqrt{5} \right) \sim 1.6 \)

The method is slower than Newton's method, but cheaper since one does not require \( f'(x) \).
§ 1.6. Bisection method

$f : [a, b] \rightarrow \mathbb{R}$, $f \in [a, b]$, $f(0) = 0$, $f$ continuous.

\[ a_0 = a, \quad a_n \rightarrow b \]

Check if there is a sign change between:

\[ a_n, c, c, b_0 \]

\[ (a_{k+1}, b_{k+1}) = \begin{cases} (a_k, c_k) \text{ if } f(a_k) f(c_k) < 0 \\ (c_k, b_k) \text{ else} \end{cases} \]

- Simply, robust
- Slower than Newton's method, at least asymptotically.
- Issues if there are several roots of $f$.

§ 1.7. Global behavior of Newton's method

Convergence if starting point $x_0$ is close to $\gamma$; general behavior can be very complicated.

Example: $f(x) = x(x^2-1)(x-3) \exp\left(-\frac{1}{2}(x-1)^2\right)$

Roots:

\[ x = 0 \]
\[ x = \pm 1 \]
\[ x = 3 \]
Newton's method for systems of equations

\[ f: \mathbb{R}^2 \to \mathbb{R}^2 \quad f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} \]

Look for \((x_0, y_0) \in \mathbb{R}^2 : \quad f(x_0, y_0) = 0\)

Newton's method: Choose \((x_0) \in \mathbb{R}^2\) initialization

\[
\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \end{pmatrix} - Df(x_k, y_k)^{-1} f(x_k, y_k) \quad k = 0, 1, 2, \ldots
\]

\[ Df(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \in \mathbb{R}^{2 \times 2} \]

Example: 
\[ f(x, y) = \begin{pmatrix} x^2 + y^4 + 2 \\ 2x^2 - y^2 - 1 \end{pmatrix} = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} \]
solution is \((x, y) = (1, 1)\), also: \((-1, 1), (1, -1)\),

\[
Df(x, y) = \begin{pmatrix}
2x & 2y \\
4x & -2y
\end{pmatrix}
\]

Newton's Start \((x_0, y_0)\)

\[
\begin{pmatrix}
x_{k+1} \\
y_{k+1}
\end{pmatrix} = \begin{pmatrix}
x_k \\
y_k
\end{pmatrix} - \begin{pmatrix}
2x_k & 2y_k \\
4x_k & -2y_k
\end{pmatrix}^{-1} \begin{pmatrix}
x_k^2 + y_k^2 - 2 \\
x_k^2 - y_k^2 - 1
\end{pmatrix}
\]

Remarks:

- requires the Jacobian matrix ("expensive")
- requires matrix inverse
  (or to solve a linear system)
- similar convergence results hold, i.e.,
  "quadratic convergence close to solution"