Joint Blind Deconvolution and Blind Demixing via Nonconvex Optimization

Shuyang Ling

Department of Mathematics, UC Davis

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Research in collaboration with:

- Prof. Xiaodong Li (UC Davis)
- Prof. Thomas Strohmer (UC Davis)
- Dr. Ke Wei (UC Davis)

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Outline

(a) Blind deconvolution meets blind demixing: applications in image processing and wireless communication

(b) Mathematical models and convex approach

(c) A nonconvex optimization approach towards joint blind deconvolution and blind demixing
What is blind deconvolution?

Suppose we observe a function $y$ which consists of the convolution of two unknown functions, the blurring function $f$ and the signal of interest $g$, plus noise $w$. How to reconstruct $f$ and $g$ from $y$?

$$y = f * g + w.$$ 

It is obviously a highly ill-posed bilinear inverse problem...

- Much more difficult than ordinary deconvolution...but have important applications in various fields.
- Solvability? What conditions on $f$ and $g$ make this problem solvable?
- How? What algorithms shall we use to recover $f$ and $g$?
Why do we care about blind deconvolution?

Image deblurring

Let $f$ be the blurring kernel and $g$ be the original image, then $y = f \ast g$ is the blurred image.

Question: how to reconstruct $f$ and $g$ from $y$
Blind deconvolution meets blind demixing

Suppose there are $s$ users and each of them sends a message $x_i$, which is encoded by $C_i$, to a common receiver. Each encoded message $g_i = C_i x_i$ is convolved with an unknown impulse response function $f_i$. 
Consider the model:

\[ y = \sum_{i=1}^{s} f_i \ast g_i + w. \]

This is even more difficult than blind deconvolution \((s = 1)\), since this is a “mixture” of blind deconvolution problems. It also includes phase retrieval as a special case if \(s = 1\) and \(\bar{g}_i = f_i\).

**More assumptions**

- Each impulse response \(f_i\) has maximum delay spread \(K\) (compact support):
  \[ f_i(n) = 0, \quad \text{for } n > K, \quad f_i = \begin{bmatrix} h_i \\ 0 \end{bmatrix}. \]

- Let \(g_i := C_i x_i\) be the signal \(x_i \in \mathbb{C}^N\) encoded by \(C_i \in \mathbb{C}^{L \times N}\) with \(L > N\). We also require \(C_i\) to be mutually incoherent by imposing randomness.
Subspace assumption on the frequency domain

Denote $F$ as the $L \times L$ DFT matrix.

- Let $h_i \in \mathbb{C}^K$ be the first $K$ nonzero entries of $f_i$ and $B$ be a low-frequency DFT matrix. There holds, $\hat{f}_i = Ff_i = Bh_i$.
- Let $\hat{g}_i := A_i x_i$ where $A_i := FC_i$ and $x_i \in \mathbb{C}^N$.

Mathematical model

$$y = \sum_{i=1}^{s} \text{diag}(Bh_i)A_i x_i + w.$$  

Goal: We want to recover $(h_i, x_i)_{i=1}^{s}$ from $(y, B, A_i)_{i=1}^{s}$.

Remark: The degree of freedom for unknowns: $s(K + N)$; number of constraints: $L$. 
Mathematical model

### Subspace assumption on the frequency domain

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- Let $\mathbf{h}_i \in \mathbb{C}^K$ be the first $K$ nonzero entries of $\mathbf{f}_i$ and $\mathbf{B}$ be a low-frequency DFT matrix. There holds, $\hat{\mathbf{f}}_i = \mathbf{Ff}_i = \mathbf{Bh}_i$.
- Let $\hat{\mathbf{g}}_i := \mathbf{A}_i \mathbf{x}_i$ where $\mathbf{A}_i := \mathbf{FC}_i$ and $\mathbf{x}_i \in \mathbb{C}^N$.

### Mathematical model

$\mathbf{y} = \sum_{i=1}^{s} \text{diag}(\mathbf{Bh}_i) \mathbf{A}_i \mathbf{x}_i + \mathbf{w}$.

**Goal:** We want to recover $(\mathbf{h}_i, \mathbf{x}_i)_{i=1}^{s}$ from $(\mathbf{y}, \mathbf{B}, \mathbf{A}_i)_{i=1}^{s}$.

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Shuyang Ling (UC Davis)  
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8 / 28
Naive approach

Nonlinear least squares

We may want to try nonlinear least squares approach:

\[
\min_{(h_i, x_i)} \left\| \sum_{i=1}^{s} \text{diag}(B h_i) A_i x_i - y \right\|_2^2.
\]

\[F(h_i, x_i)\]

- The objective function is highly nonconvex and more complicated than blind deconvolution \((s = 1)\).
- Gradient descent might get stuck at local minima.
- No guarantees for recoverability.
Naive approach

Nonlinear least squares

We may want to try nonlinear least squares approach:

$$\min_{(h_i, x_i)} \left\| \sum_{i=1}^{s} \text{diag}(B h_i) A_i x_i - y \right\|^2_{F(h_i, x_i)}.$$  

- The objective function is highly nonconvex and more complicated than blind deconvolution ($s = 1$).
- Gradient descent might get stuck at local minima.
- No guarantees for recoverability.
Let \( a_{i,l} \) be the \( l \)-th column of \( A_i^* \) and \( b_l \) be the \( l \)-th column of \( B^* \).

\[
y_l = \sum_{i=1}^{s} (B h_i)_l \cdot (A_l x_i)_l = \sum_{i=1}^{s} b_l^* h_i x_i^* a_{i,l}.
\]

Let \( X_i := h_i x_i^* \) and define the linear operator \( A_i : \mathbb{C}^{K \times N} \rightarrow \mathbb{C}^{L} \) as,

\[
A_i(Z) := \left\{ b_l^* Z a_{i,l} \right\}_{l=1}^{L} = \left\{ \langle Z, b_l a_{i,l}^* \rangle \right\}_{l=1}^{L}.
\]

Then, there holds \( y = \sum_{i=1}^{s} A_i(X_i) + w \).

See [Candès-Strohmer-Voroninski 13], [Ahmed-Recht-Romberg, 14].
Convex relaxation and low-rank matrix recovery

**Rank-s matrix recovery**

We rewrite $y = \sum_{i=1}^{s} \text{diag}(B_h)A_i x_i$ as

$$y_l = \begin{bmatrix} h_1x_1^* & 0 & \cdots & 0 \\ 0 & h_2x_2^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_sx_s^* \end{bmatrix}, \begin{bmatrix} b_{1,1}^* \\ 0 \\ \vdots \\ 0 \\ b_{s,1}^* \end{bmatrix}, \begin{bmatrix} 0 & \cdots & 0 \\ 0 & b_{1,2}^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{s,2}^* \end{bmatrix}$$

- Recover a **rank-s block diagonal** matrix satisfying convex constraints.
- Finding such a rank-s matrix is generally an NP-hard problem.
Low-rank matrix recovery

Nuclear norm minimization

The ground truth is a rank-$s$ block-diagonal matrix. It is natural to recover the solution via solving

$$\min \sum_{i=1}^{s} \|Z_i\|_* \quad \text{subject to} \quad \sum_{i=1}^{s} A_i(Z_i) = y$$

where $\sum_{i=1}^{s} \|Z_i\|_*$ is the nuclear norm of $\text{blkdiag}(Z_1, \cdots, Z_s)$.

Question: Can we recover $\{h_0x_{i0}^*\}_{i=1}^{s}$ exactly?
Theorem

Assume that

\( \text{Let } B \in \mathbb{C}^{L \times K} \text{ be a partial DFT matrix with } B^* B = I_K; \)

\( \text{Each } A_i \text{ is a Gaussian random matrix.} \)

The SDP relaxation is able to recover \( \{(h_{i0}, x_{i0})\}_{i=1}^{s} \) exactly with probability at least \( 1 - \mathcal{O}(L^{-\gamma}) \). Here the number of measurements \( L \) satisfies

\( \text{[Ling-Strohmer 15]} \quad L \geq C_\gamma s^2 (K + \mu_h^2 N) \log^3 L; \)

\( \text{[Jung-Krahmer-Stöger 17]} \quad L \geq C_\gamma (s(K + \mu_h^2 N)) \log^3 L \)

where \( \mu_h^2 = L \max_{1 \leq i \leq s} \frac{\|Bh_{i0}\|_\infty^2}{\|h_{i0}\|_2^2}. \)

We can jointly estimate the channels and signals for \( s \) users with one simple convex program.

SDP is able to recover \( \{(h_i, x_i)\}_{i=1}^{s} \) but it is computationally expensive.
Convex approach

Theorem

Assume that

- Let $B \in \mathbb{C}^{L \times K}$ be a partial DFT matrix with $B^* B = I_K$;
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The SDP relaxation is able to recover $\{(h_{i0}, x_{i0})\}_{i=1}^s$ exactly with probability at least $1 - \mathcal{O}(L^{-\gamma})$. Here the number of measurements $L$ satisfies

- [Ling-Strohmer 15] $L \geq C_\gamma s^2 (K + \mu_h^2 N) \log^3 L$;
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- We can jointly estimate the channels and signals for $s$ users with one simple convex program.
- SDP is able to recover $\{(h_i, x_i)\}_{i=1}^s$ but it is computationally expensive.
A nonconvex optimization approach?

An increasing list of nonconvex approaches to various problems in machine learning and signal processing:

- Phase retrieval: Candès, Li, Soltanolkotabi, Chen, Wright, Sun, etc...
- Matrix completion: Sun, Luo, Montanari, etc...
- Various problems: Recht, Wainwright, Constantine, etc...

Two-step philosophy for provable nonconvex optimization

(a) Use spectral method to construct a starting point inside “the basin of attraction”;

(b) Run gradient descent method.

The key is to build up “the basin of attraction”.
The basin of attraction relies on the following three observations.

**Observation 1: Unboundedness of solution**

- If the pair \((h_{i0}, x_{i0})\) is a solution to \(y = \sum_{i=1}^{s} \text{diag}(Bh_{i0})A_i x_{i0}\), then so is the pair \((\alpha_i h_{i0}, \alpha_i^{-1} x_{i0})\) for any \(\alpha_i \neq 0\).

- Thus the blind deconvolution problem always has infinitely many solutions of this type. We can recover \((h_{i0}, x_{i0})\) only up to a scalar.

- It is possible that \(\|h_i\| \gg \|x_i\|\) (vice versa) while \(\|h_i\| \cdot \|x_i\|\) is fixed. Hence we define \(\mathcal{N}_{d_0}\) to balance \(\|h_i\|\) and \(\|x_i\|\):

\[
\mathcal{N}_{d_0} := \\left\{ ((h_i, x_i))_{i=1}^{s} : \|h_i\| \leq 2\sqrt{d_{i0}}, \|x_i\| \leq 2\sqrt{d_{i0}} \right\}.
\]

where \(d_{i0} = \|h_{i0}\| \cdot \|x_{i0}\|\).
Observation 2: Incoherence

Our numerical experiments have shown that the algorithm’s performance depends on how much $b_l$ (the rows of $B$) and $h_{i0}$ are correlated.

$$\mu_h^2 := \max_{1 \leq i \leq s} \frac{L \| Bh_{i0} \|_\infty^2}{\| h_{i0} \|_2^2}, \quad \text{the smaller } \mu_h, \text{ the better.}$$

Therefore, we introduce the $\mathcal{N}_\mu$ to control the incoherence:

$$\mathcal{N}_\mu := \{ \{ h_i \}_{i=1}^s : \sqrt{L} \| Bh_i \|_\infty \leq 4\mu \sqrt{d_{i0}} \}.$$  

“Incoherence” is not a new idea. In matrix completion, we also require the left and right singular vectors of the ground truth cannot be too “aligned” with those of measurement matrices $\{ b_l a_{i,l}^* \}_{1 \leq l \leq L}$.
Observation 3: “Close” to the ground truth

We define $\mathcal{N}_\varepsilon$ to quantify closeness of $\{(h_i, x_i)\}^s_{i=1}$ to true solution, i.e.,

$$\mathcal{N}_\varepsilon := \{(h_i, x_i)\}^s_{i=1} : \|h_i^* x_i^* - h_i^0 x_i^0\|_F \leq \varepsilon d_i^0 \}.$$ 

We want to find an initial guess close to $\{(h_i^0, x_i^0)\}^s_{i=1}$. 

Building “the basin of attraction”
Based on the three observations above, we define the three neighborhoods:

**The basin of attraction**

The basin of attraction is the intersection of the following three sets \( \mathcal{N}_{d_0} \cap \mathcal{N}_\mu \cap \mathcal{N}_\varepsilon \):

\[
\mathcal{N}_{d_0} := \left\{ \left\{ (\mathbf{h}_i, \mathbf{x}_i) \right\}_{i=1}^s : \|\mathbf{h}_i\| \leq 2\sqrt{d_{i0}}, \|\mathbf{x}_i\| \leq 2\sqrt{d_{i0}}, 1 \leq i \leq s \right\}
\]

\[
\mathcal{N}_\mu := \left\{ \{ \mathbf{h}_i \}_{i=1}^s : \sqrt{L} \| \mathbf{Bh}_i \|_\infty \leq 4\sqrt{d_{i0}\mu}, 1 \leq i \leq s \right\}
\]

\[
\mathcal{N}_\varepsilon := \left\{ \left\{ (\mathbf{h}_i, \mathbf{x}_i) \right\}_{i=1}^s : \frac{\|\mathbf{h}_i\mathbf{x}_i^* - \mathbf{h}_{i0}\mathbf{x}_{i0}^*\|_F}{d_{i0}} \leq \varepsilon, 1 \leq i \leq s \right\}
\]

where \( d_{i0} = \|\mathbf{h}_{i0}\|\|\mathbf{x}_{i0}\| \), \( \mu \) is a parameter and \( \mu \geq \mu_h \) and \( \varepsilon \) is a predetermined parameter in \((0, \frac{1}{15}]\).
Objective function: a variant of projected gradient descent

The objective function \( \tilde{F} \) consists of two parts: \( F \) and \( G \):

\[
\min_{(h, x)} \tilde{F}(h, x) := F(h, x) + G(h, x)
\]

least squares term

regularization term

where \( F(h, x) := \|\sum_{i=1}^{s} A_i(h; x_i^*) - y\|^2 \) and

\[
G(h, x) := \rho \sum_{i=1}^{s} \left[ G_0 \left( \frac{\|h_i\|^2}{2d_i} \right) + G_0 \left( \frac{\|x_i\|^2}{2d_i} \right) + \sum_{l=1}^{L} G_0 \left( \frac{L|b_i^* h_i|^2}{8d_i \mu^2} \right) \right].
\]

\( \mathcal{N}_{d_0} \): balance \( \|h_i\| \) and \( \|x_i\| \)

\( \mathcal{N}_{\mu} \): impose incoherence

Here \( G_0(z) = \max\{z - 1, 0\}^2 \), \( \rho \approx d^2 \), \( d \approx d_0 \), \( d_i \approx d_{i0} \) and \( \mu \geq \mu_h \).
Algorithm: Initialization via spectral method

Note that

$$A_i^*(y) = \underbrace{A_i^* A_i(h_{i0}x_{i0}^*)}_{\mathbb{E}(A_i^* A_i(h_{i0}x_{i0}^*)) = h_{i0}x_{i0}^*} + A_i^* \left( \sum_{j \neq i} A_j(h_{j0}x_{j0}^*) \right)$$

with mean 0

The leading singular vectors of $A_i^*(y)$ can approximate $(h_{i0}, x_{i0})$.

**Step 1: Initialization via spectral method and projection:**

1. **for** $i = 1, 2, \ldots, s$ **do**
2. Compute $A_i^*(y)$, (since $\mathbb{E}(A_i^*(y)) = h_{i0}x_{i0}^*$);
3. $(d, \hat{h}_{i0}, \hat{x}_{i0}) = \text{svds}(A_i^*(y))$;
4. $u_{i}^{(0)} := \mathcal{P}_{\mathcal{N}_\mu}(\sqrt{d_i}\hat{h}_{i0})$ and $v_{i}^{(0)} := \sqrt{d_i}\hat{x}_{i0}$.
5. **end for**
Algorithm: Wirtinger gradient descent

Step 2: Gradient descent with constant stepsize $\eta$:

1: **Initialization:** obtain $(u^{(0)}_i, v^{(0)}_i)$ via Algorithm 1.
2: for $t = 1, 2, \ldots, \text{do}$
3: for $i = 1, 2, \ldots, s \text{ do}$
4: \quad $u_i^{(t)} = u_i^{(t-1)} - \eta \nabla \tilde{F}_{h_i}(u^{(t-1)}, v^{(t-1)})$
5: \quad $v_i^{(t)} = v_i^{(t-1)} - \eta \nabla \tilde{F}_{x_i}(u^{(t-1)}, v^{(t-1)})$
6: end for
7: end for
Main results

Theorem [Ling-Strohmer 17]

Assume \( w \sim \mathcal{CN}(0, \sigma^2 d_0^2 / L) \) and \( A_i \) as a complex Gaussian matrix. There hold:

- the initial guess \((u^{(0)}, v^{(0)})\) \(\in\) \(\sqrt{3}\mathcal{N}_{d_0} \cap \frac{1}{\sqrt{3}}\mathcal{N}_\mu \cap \mathcal{N}_{\frac{2\varepsilon}{5\sqrt{s\kappa}}}\),

\[
\sqrt{\sum_{i=1}^{s} \| u_i(t)(v_i(t))^* - h_{i0} x_{i0}^* \|_F^2} \leq (1 - \alpha)^t \varepsilon d_0 + c_0 \sqrt{s} \| A^*(w) \|
\]

with probability at least \(1 - L^{-\gamma + 1}\) and \(\alpha = O((s(K + N) \log^2 L)^{-1})\) if

\[
L \geq C_\gamma (\mu_h^2 + \sigma^2)s^2 \kappa^4 (K + N) \log^2 L \log s / \varepsilon^2,
\]

where \(\kappa = \frac{\max d_{i0}}{\min d_{i0}}\).
Main results

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Assume $\mathbf{w} \sim \mathcal{CN}(0, \sigma^2 d_0^2 / L)$ and $\mathbf{A}_i$ as a complex Gaussian matrix. There hold:

- the initial guess $(\mathbf{u}^{(0)}, \mathbf{v}^{(0)}) \in \frac{1}{\sqrt{3}} \mathcal{N}_{d_0} \cap \frac{1}{\sqrt{3}} \mathcal{N}_\mu \cap \mathcal{N}_{\frac{2\varepsilon}{5\sqrt{s}\kappa}}$,

- $\sqrt{\sum_{i=1}^{s} \| \mathbf{u}_i^{(t)}(\mathbf{v}_i^{(t)})^* - \mathbf{h}_{i0} \mathbf{x}_{i0}^* \|_F^2} \leq \left(1 - \alpha\right)^t \varepsilon d_0 + c_0 \sqrt{s} \| \mathbf{A}^*(\mathbf{w}) \|

with probability at least $1 - L^{-\gamma + 1}$ and $\alpha = \mathcal{O}((s(K + N) \log^2 L)^{-1})$ if

$$L \geq C_\gamma (\mu_h^2 + \sigma^2) s^2 \kappa^4 (K + N) \log^2 L \log s / \varepsilon^2,$$

where $\kappa = \frac{\max d_{i0}}{\min d_{i0}}$. 

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Remark

- The iterates \((u_i^{(t)}, v_i^{(t)})\) converges linearly to \((h_{i0}, x_{i0})\):

\[\|u_i^{(\infty)}(v_i^{(\infty)})^* - h_{i0}x_{i0}\|_F \leq c_0 \sqrt{s} \|A^*(w)\|\]

- \(\|A^*(w)\|\) converges to 0 with the rate of \(O(L^{-1/2})\):

\[\|A^*(w)\| \leq C_0 \sigma d_0 \sqrt{s(K + N)(\log^2 L)/L}\]

Therefore, \((u_i^{(\infty)}, v_i^{(\infty)})\) is a consistent estimator of \((h_{i0}, x_{i0})\).

- Challenges: \(s^2\) is not optimal. The optimal scaling should be \(L = O(s(K + N))\) instead of \(L = O(s^2(K + N))\).
Remark

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- \(\|A^*(w)\|\) converges to 0 with the rate of \(O(L^{-1/2})\):
  \[
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  \]

Therefore, \((u_i^{(\infty)}, v_i^{(\infty)})\) is a consistent estimator of \((h_{i0}, x_{i0})\).

- Challenges: \(s^2\) is not optimal. The optimal scaling should be \(L = O(s(K + N))\) instead of \(L = O(s^2(K + N))\).
Let each $A_i$ be a complex Gaussian matrix. The number of measurement scales linearly with the number of sources $s$ if $K$ and $N$ are fixed. Approximately, $L \approx 1.5s(K + N)$ yields exact recovery.

**Figure:** Black: failure; white: success
Back to the communication example

A more practical and useful choice of encoding matrix $C_i$: $C_i = D_iH$ (i.e., $A_i = F D_iH$) where $D_i$ is a diagonal random binary $\pm 1$ matrix and $H$ is an $L \times N$ deterministic partial Hadamard matrix. With this setting, our approach can demix many users without performing channel estimation.

$L \approx 1.5s(K + N)$ yields exact recovery.
We see that the relative error is linearly correlated with the noise in dB. Approximately, 10 units of increase in SNR leads to the same amount of decrease in relative error (in dB).
Conclusion: The proposed algorithm is arguably the first blind deconvolution/blind demixing algorithm that is numerically efficient, robust against noise and comes with rigorous recovery guarantees under subspace conditions.

- Open problem: Does similar result hold for other types of $A_i$?
- Open problem: what if either $h_i$ or $x_i$ is sparse?
- Major open problem in nonconvex optimization:
  How to remove the $s^2$-dependence for rank-$s$ matrix recovery?
**Conclusion:** The proposed algorithm is arguably the first blind deconvolution/blind demixing algorithm that is numerically efficient, robust against noise and comes with rigorous recovery guarantees under subspace conditions.

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- **Major open problem in nonconvex optimization:**
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