VERTICAL STRUCTURE OF POTENTIAL VORTICITY FLUXES IN WEAKLY
(AND STRONGLY) UNSTABLE FLOWS

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1. INTRODUCTION

Eddying flow in the ocean develops as a result of baroclinic instability, and the environment in which the flow develops is vertically and horizontally inhomogeneous. A theory that takes into account the vertical inhomogeneities and predicts the magnitude and structure of the eddy potential vorticity flux for strongly baroclinically unstable mean shears is reviewed and extended to include the full range of supercriticalities. The theory applies equally to systems with uniform and non-uniform stratification and utilizes the neutral stratification modes and the projection of the mean shear onto these modes. For both strongly and weakly unstable shears, the predicted eddy potential vorticity flux automatically conserves momentum. In the weakly unstable case this result is non-trivial; because β contributes non-negligibly to the mean potential vorticity gradient, the potential vorticity flux can no longer be simply proportional, at each level, to the vortex-stretching part of the potential vorticity gradient and still conserve momentum.

2. FLOW SCALES

In the weakly unstable limit (WUL), the flow is unstable, yet there is little scale separation between the first deformation scale and the barotropic halting scale, hence not much of an inverse cascade. In this case we might estimate the mixing length, or "halting scale" of the cascade as the deformation scale, *i.e.,*

$$k_{\mathrm{mix}} \simeq \lambda_1. \tag{1}$$

The feedback that exists between the halting scale and the generation rate noted by Held and Larichev (1996) in the strongly unstable case is

now avoided. One should not confuse the above statement with the theory of Stone (1972), who posed the same mixing scale for all eddies generated by baroclinic instability.

The results of Smith et al. (2002, hereafter SB) suggest that when both linear drag and β are present, the mixing scale and the largest barotropic energy scale may not be the same. They present simulations of two-dimensional turbulence forced at small scale, and modified by linear drag and β. In addition, a tracer, with a fixed large-scale mean gradient, is stirred by this flow. In those simulations, jets form at strengths and widths which are combined functions of drag and β. In contrast, the mixing of the tracer occurred at a scale which depended only on β, not on drag.

We can extend the argument of SB to the baroclinic case. The spatially averaged energy budget for the barotropic mode in the presence of linear drag can be written

$$\frac{dE_0}{dt} = -2rE_0 + G_0(t) \tag{2}$$

where $\Gamma_0(t)$ represents all the non-zero baroclinic-barotropic transfer terms (which act to force the barotropic mode) less the small-scale dissipation. Since there is no net barotropic eddy generation, the transfer term must equal the baroclinic energy generation rate. SB found that the stirring of tracer, forced by a large scale mean meridional gradient (hence analogous to baroclinic potential vorticity at large scale), occurred at the scale $k_\beta \simeq (\beta^{3/5}/\epsilon)^{1/5}$, termed the "inviscid Rhines" scale, despite that jets formed at a larger scale that depends on a combination of drag and β.

We assume that the stirring of the baroclinic potential vorticity, or the baroclinic energy generation rate, occurs at k_β, and is largely unaffected by drag, and we write the generation, dissipation

and barotropic energy such that

$$G_0(t) = g_0 + \gamma(t), \quad \langle G_0 \rangle = g_0 \qquad (3)$$
$$E_0(t) = e_0 + \epsilon(t), \quad \langle E_0 \rangle = e_0, \qquad (4)$$

with a time average defined as

$$\langle f(t) \rangle \equiv \frac{1}{\tau} \int_t^{t+\tau} \frac{df}{dt'} \, dt'.$$

Then

$$\left\langle \frac{dE_0}{dt} \right\rangle = \frac{\epsilon(t+\tau) - \epsilon(t)}{\tau} = -2re_0 + g_0.$$

If the flow is in statistically steady state, then $[\epsilon(t+\tau) - \epsilon(t)]/\tau \to 0$, for all sufficiently large τ (say, larger than an eddy turn-around time), and

$$e_0 \simeq \frac{g_0}{2r}. \qquad (5)$$

If the drag is small enough that anisotropy is allowed to develop, yet the cascade is halted by β, then the barotropic energy will consist of large-scale zonal jets, and $e_0 \simeq U_0^2/2$, so that $U_0 \simeq (g_0/r)^{1/2}$.

SB (and references) find that the anisotropic cascade occurs among the meridional wavenumbers, and forms a k^{-5} spectrum. Setting the integral of the energy spectrum considered by SB to the total energy found in (5),

$$\int_{k_{\beta,r}}^{\infty} \mathcal{C}_\beta \beta^2 k^{-5} \, dk \simeq \frac{g_0}{2r} \qquad (6)$$

leads to

$$k_{\beta,r} \simeq \left(\frac{\mathcal{C}_\beta \beta^2 r}{2g_0} \right)^{1/4}. \qquad (7)$$

Smith and Vallis (2002, hereafter SV) find, in a series of five-layer runs in which the first baroclinic mode was forced and bottom drag was varied, that the generation had a weak dependence on drag, $\epsilon \sim r^{-1/4}$. Substituting this dependence in (7) alters the dependence of $k_{\beta,r}$ from $r^{1/4}$ to $r^{5/16}$ — likely an undetectable difference.

The inviscid β scale is the scale discussed by Maltrud and Vallis (1991),

$$k_\beta = \beta^{3/5} \epsilon^{-1/5}, \qquad (8)$$

and it is at this scale that we expect baroclinic eddy generation to occur. In the weakly unstable limit, this scale is nearly coincident with the deformation.

Barotropic energy accumulates at $k_y = \pm k_{\beta,r}$ in the two-dimensional spectral plane, with phases that are no longer randomized by strong turbulence, but that are approximately set by the Rossby wave dispersion relation. These sharp peaks also cover a very limited 'area' in the wavenumber plane, and correspond to zonal jets. Thus it is not surprising that, despite its large amplitude, energy at the viscous β scale contributes negligibly to the variance-generating correlation between the meridional eddy velocity and the baroclinic PV.

3. FLOW ENERGIES

Assuming a weak, but existant, inverse barotropic cascade, most of the arguments in SV can be carried over to the present case. Using their estimate equation (B.3) for the baroclinic eddy energy generation rate for N modes,

$$g_0 \simeq \sum_{m=1}^{N} \bar{U}_m \lambda_m^2 \overline{V'_m \Psi'_0}, \qquad (9)$$

and their estimate the baroclinic eddy velocities at the mixing scale,

$$\langle V_m \rangle_{k_{\mathrm{mix}}} \simeq \bar{U}_m, \qquad (10)$$

we can express the generation rate as

$$g_0 \simeq V_0 k_{\mathrm{mix}}^{-1} \mu^2 \qquad (11)$$

where

$$\mu^2 \equiv T_e^{-2} = \sum_{m=1}^{N} \bar{U}_m^2 \lambda_m^2. \qquad (12)$$

A departure from the theory presented in SV must be made in relating the generation rate to the barotropic velocity. The previous section implies that the relevant barotropic velocity is the isotropic meridional eddy velocity at the mixing scale k_{mix}, which is not necessarily coincident with the drag-sensitive, anisotropic peak. Following equation (7.17) of SB, we estimate the isotropic

mixing velocity (the isotropic barotropic meridional velocity magnitude at the mixing scale k_{mix}) as

$$V_{\mathrm{mix}} \simeq \left[\int_{k_{\mathrm{mix}}}^{\lambda} \mathcal{C}\epsilon^{2/3}k^{-5/3}\,dk\right]^{1/2}$$
$$= \left[\frac{3}{2}\mathcal{C}\epsilon^{2/3}(k_{\mathrm{mix}}^{-2/3} - \lambda_1^{-2/3})\right]^{1/2}, \quad (13)$$

where ϵ is the energy transfer rate, equal to the baroclinic generation g_0. In the WUL, if we truly take $k_{\mathrm{mix}} = \lambda_1$, then $V_{\mathrm{mix}} = 0$. In the formal sense, this is reasonable if we expect no mixing or generation without baroclinic instability, and we will make this assumption[1]. Eliminating g_0 between (13) and (11) yields

$$V_{\mathrm{mix}} \simeq (3\mathcal{C}/2)^{3/4}\mu k_{\mathrm{mix}}^{-1}[1 - (k_{\mathrm{mix}}/\lambda_1)^{2/3}]^3, \quad (14)$$
$$g_0 \simeq (3\mathcal{C}/2)^{3/4}\mu^3 k_{\mathrm{mix}}^{-2}[1 - (k_{\mathrm{mix}}/\lambda_1)^{2/3}]^3. \quad (15)$$

If we use (8) as an estimate for k_{mix}, then a feedback between the generation rate and the stopping scale will be introduced. The resulting equations cannot be solved directly, so we take an iterative solution. Taking the limit $k_{\mathrm{mix}}/\lambda_1 \to 0$ in (15) and setting $k_{\mathrm{mix}} = k_\beta$ and $\epsilon = g_0$ in (8), we get the solution valid in the strongly unstable limit, $k_{\mathrm{mix}} \simeq \alpha^{1/4}\mu^{-1}\beta$, where $\alpha = 3\mathcal{C}/2$. Substituting this directly into (15) then gives the iterated solution for the barotropic mixing velocity

$$V_{\mathrm{mix}} \simeq \alpha\mu^2\beta^{-1}(1 - \Delta^{2/3})^3. \quad (16)$$

where $\Delta = \alpha^{1/4}\beta(\mu\lambda_1)^{-1}$ is nearly equal to the inverse of the two-layer supercriticality.

4. POTENTIAL VORTICITY FLUX

Following SV we can expand the horizontally averaged potential vorticity flux in vertical modes

$$\overline{v'q'} = \sum_{m,n=0}^{N} a_{mn}\phi_m(z)\phi_m(z) \quad (17)$$

[1]One should note the following caveats. First, linear drag can induce baroclinic generation, even in the inviscid stable limit, and further, linear dynamics with drag can mix a tracer in much the same way Brownian motion leads to net particle displacements. Second, when the mean shear and/or stratification are surface intensified, the system may be baroclinically unstable at significantly smaller scale than the first deformation scale.

where

$$a_{mn} = \overline{V'_m Q'_n} \quad (18)$$

Estimating the modal coefficients in the summation is useful for two reasons. First, because cascade processes occur in scale rather than space, turbulence phenomenology leads us to estimates of the coefficients more directly than to direct estimates of $\overline{v'q'}$ at each level. Second, because we know that $a_{mm} = 0$ for all m (see SV), and because the vertical modes $\phi_m(z)$ are orthonormal, estimates of the modal coefficients a_{mn} will leave us with an estimate for the vertical structure of the flux whose vertical integral vanishes. This constraint is necessary for momentum conservation in the TEM.

Potential vorticity in the neighborhood of the mixing scale can be estimated as

$$\langle Q'_m \rangle \simeq -(k_{\mathrm{mix}}^2 + \lambda_m^2)V_m k_{\mathrm{mix}}^{-1}, \quad (19)$$

where V_m is an estimate of the relevant velocity scale for mode m. For $m = 0$, (16) is our estimate, and for $m > 0$, we use (10). Combining the above results, we propose the estimate

$$a_{mn} \simeq -V_m(k_{\mathrm{mix}}^2 + \lambda_n^2)k_{\mathrm{mix}}^{-1}V_n(1 - \delta_{mn}), \quad (20)$$

with

$$V_m = \begin{cases} \alpha\mu^2\beta^{-1}(1 - \Delta^{2/3})^3, & m = 0, \\ |\bar{U}_m|, & m > 0. \end{cases} \quad (21)$$

A key part of this formulation is that, in the WUL, the rms barotropic eddy velocity is not necessarily large in comparison to the rms baroclinic velocities, so we cannot neglect terms like $a_{m0} = \overline{V'_m Q'_0}$ (baroclinic advection of barotropic PV). The present form does reduce to the form suggested by SV when Δ is small.

5. NUMERICAL TEST

Figure 1 shows comparisons of the theory for the northward eddy potential vorticity flux to that obtained from simulations using a horizontally homogeneous QG model and realistic profiles of stratification and zonal mean shear. The left panel is

similar to SV figure 15 and is based on a continuation of the same strongly unstable, 15-layer simulation reported therein (see SV for a more complete description). The true steady-state now obtained for that simulation reveals that the theory of SV fits the simulated data without multiplication by an overall scale factor. The fit is not perfect, notably near the surface, where the predicted southward surface flux far overshoots the simulated result. The right panel shows results from a recent 10-layer simulation using weakly unstable mean flow and non-uniform profiles of stratification and mean shear. The profiles were derived from data taken for the North Atlantic Tracer Release Experiment in the eastern Atlantic ocean. The theory is apparently even better at predicting the PV flux magnitude and structure in the weakly unstable limit.

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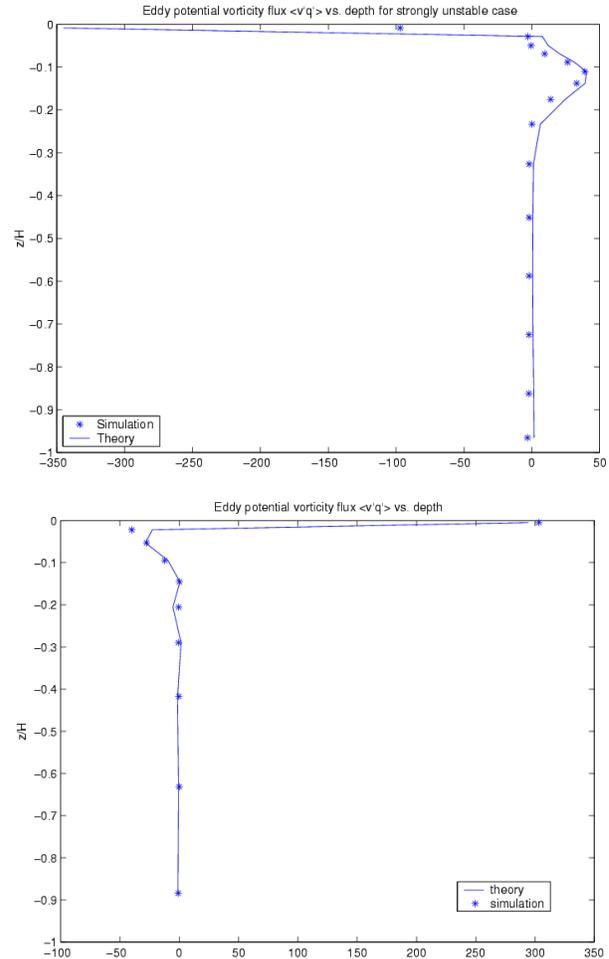
Figure 1: Top: profile of northward eddy potential vorticity flux $\overline{v'q'}$ from 256^2 by 15-layer simulation (asterisks) using mean profiles that are interpolations from the shear and potential density profiles generated by a primitive equation simulation of the North Atlantic, compared to theory based on (17) with (20) and (21) (solid). Bottom: profile of northward eddy potential vorticity flux $\overline{v'q'}$ from 512^2 by 10-layer simulation (asterisks) using mean profiles taken from NATRE, compared to same theory (solid).