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Multiscale models for synoptic–mesoscale interactions in the ocean

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ABSTRACT

Multiscale analysis is used to derive two sets of coupled models, each based on the same distinguished limit, to represent the interaction of the midlatitude oceanic synoptic scale—where coherent features such as jets and rings form—and the mesoscale, defined by the internal deformation scale. The synoptic scale and mesoscale overlap at low and mid latitudes, and are hence synonymous in much of the oceanographic literature; at higher latitudes the synoptic scale can be an order of magnitude larger than the deformation scale, which motivates our asymptotic approach and our nonstandard terminology. In the first model the synoptic dynamics are described by 'Large Amplitude Geostrophic' (LAG) equations while the eddy dynamics are quasigeostrophic. This model has order one isopycnal variation on the synoptic scale; the synoptic dynamics respond to an eddy momentum flux while the eddy dynamics respond to the baroclinically unstable synoptic density gradient. The second model assumes small isopycnal variation on the synoptic scale, but allows for a planetary scale background density gradient that may be fixed or evolved on a slower time scale. Here the large-scale equations are just the barotropic quasigeostrophic equations, and the mesoscale is modeled by the baroclinic quasigeostrophic equations. The synoptic dynamics now respond to both eddy momentum and buoyancy fluxes, but the small-scale eddy dynamics are simply advected by the synoptic-scale flow—there is no baroclinic production term in the eddy equations. The energy

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budget is closed by deriving an equation for the slow evolution of the eddy energy, which ensures that energy gained or lost by the synoptic-scale flow is reflected in a corresponding loss or gain by the eddies. This latter model, aided by the eddy energy equation—a key result of this paper—provides a conceptual basis through which to understand the classic baroclinic turbulence cycle.

1. Introduction

In recent decades, extensive observations and high-resolution numerical models have revealed a rich array of coherent structures—jets, vortices and Rossby waves (e.g. Chelton et al., 2011)—on scales larger than the deformation scale but well below the scale of the gyres. Such features characterize what we will call the "oceanic synoptic scale" and have a major impact on ocean mixing and transport. Understanding their generation and maintenance is therefore necessary to refine theories of the oceanic general circulation. Moreover, accurately representing such features is a major goal of computational physical oceanography, particularly as global scale ocean climate models move into the eddy-permitting regime.

Our understanding of such features (at least, in mid- and high-latitudes) is primarily based on their ability to be modeled by the quasigeostrophic (QG) equations—equations that result from a single-scale asymptotic expansion of the primitive equations. Likewise, much of our theoretical understanding of the gyre-scale circulation is essentially grounded in the planetary geostrophic (PG) equations, which are derived in the same way but involve a different distinguished limit. The problem of understanding the generation, maintenance and rectification of the synoptic scale is ultimately a question of how the processes in these two regimes interact.

One approach to disentangling dynamical interactions across disparate scales is through the use of multiple scale asymptotics (MSA). Multiple scale asymptotic analysis has recently been applied to a number of atmospheric regimes (Majda and Klein, 2003; Majda, 2007a,b), offering insight into, for example, the hurricane embryo problem (Majda et al., 2010). See also Klein (2010) for a review of single- and multi-scale asymptotic results for the atmosphere.

For the midlatitude ocean, Pedlosky (1984) used MSA to derive QG and PG as, respectively, the small-scale, fast-time and large-scale, slow-time components of a two-scale model. Grooms et al. (2011) revisited this approach and generalized the model, delineating the conditions under which the interaction was from large-to-small, small-to-large, and two-way. In the latter case, which only occurs when the large-scale flow is anisotropic, the slowly evolving local PG mean state generates baroclinic instability in the QG model, producing eddy fluxes that feed back on the PG flow.

The PG–QG interaction models describe the connection between the gyre-scale flow, on the one hand, and the entire network of unstable mesoscale flows and the synoptic-scale features they produce on the other. There is little distinction, if any, between the oceanic synoptic scale and the mesoscale at low and mid latitudes; indeed the terms are synonymous in most of the oceanographic literature. But at higher latitudes the baroclinic deformation radius can be much smaller than the scale of the energy-containing eddies, the synoptic scale; rather than develop a new term for the scale of the baroclinic deformation radius, as distinct from the synoptic scale, we prefer to bend the standard terminology by referring to the deformation scale as the 'mesoscale'. The goal of the present work is to derive models that represent the interaction of the synoptic scale with the mesoscale—how eddy fluxes generate synoptic-scale features, and how rectified synoptic flows then alter the baroclinically unstable background affecting the mesoscale. In the Antarctic Circumpolar Current (ACC), for example, the baroclinic deformation radius is between 10 and 20 km (Chelton et al., 1998: Fig. 6) while the mean eddy scale is between 60 and 90 km (Chelton et al., 2011: Fig. 12); the ratio of these scales, between 1/3 and 1/9, suggests an asymptotic approach.

Setting the large scale equal to the synoptic scale—smaller than the planetary scale but larger than the deformation scale—and the small scale equal to the deformation radius, two possible limits arise.

In the first, we consider a mean flow defined by $O(1)$ isopycnal variations on the synoptic scale. This parameter regime is described by the 'Large Amplitude Geostrophic' (LAG) equations (Benilov, 1993). In its original derivation, LAG arises from single-scale asymptotic analysis, in a third distinguished limit between PG and QG. In the multiscale approach, the LAG equations arise naturally as the mean model, with QG as the small-scale dynamics.

In the second limit, we assume small isopycnal variations on the synoptic scale. In contrast to the above case, however, we here allow for a slowly evolving planetary-scale mean flow, described perhaps by the PG equations, but whose evolution we do not consider. In this case, the synoptic-scale mean flow is barotropic QG with a passive baroclinic field, and the small-scale flow is baroclinic QG. We can think of this case as a multiscale model for the barotropic–baroclinic eddy cycle model of Salmon (1980): the fixed background mean planetary flow generates baroclinic instability in the baroclinic mesoscale model, which results in an inverse cascade that alters the synoptic scale, and this subsequently alters the baroclinic instability.

An important distinction between the two models is the following. In the first model, the synoptic-scale dynamics respond only to eddy momentum fluxes in the barotropic vorticity equation—there is no eddy buoyancy flux term in the synoptic-scale buoyancy equation, thus lateral density gradients cannot be relaxed by eddy fluxes, despite that the eddies are being continually generated by baroclinic instability in the eddy equation. By contrast, in the second limit, the synoptic-scale responds to both eddy momentum and buoyancy fluxes, but now the eddy equation has no baroclinic production term. This second model, however, allows for the derivation of an equation determining the slow-time evolution of the eddy energy, which ensures that energy gained or lost by the synoptic-scale flow is reflected in a corresponding loss or gain by the eddies. This second model—appended by the eddy energy evolution equation—provides a diagnostic tool set through which synoptic–eddy interactions may be analyzed. For example, one may use the energy equation to assess the relative importance of mean advection, local production, and local dissipation in setting eddy energy at a given region in the midlatitude oceans; experimentally assessing the importance of local (dissipation, baroclinic production) and nonlocal (mean advection and wave radiation) terms in the eddy energy budget is a topic of current research to be published in a subsequent paper.

The paper is organized as follows. In Section 2 we derive the general MSA framework on which our two models are based. In Section 3 we consider the first limit, with strong synoptic-scale isopycnal gradients (the LAG–QG model), and in Section 4 we present the second limit, with weak synoptic-scale isopycnal gradients and an imposed, fixed planetary-scale gradient (the barotropic–baroclinic QG model). We discuss and conclude in Section 5.

2. Asymptotic framework for midlatitude ocean eddies

We frame our investigation in the context of the inviscid, adiabatic hydrostatic primitive equations on a β-plane

$$\partial_t \mathbf{u}_h + \mathbf{u}\cdot\nabla\mathbf{u}_h + (f_0+\beta y)\hat{\mathbf{z}}\times\mathbf{u}_h + \frac{1}{\rho_0}\nabla_h p = 0 \tag{1}$$

$$\partial_t b + \mathbf{u}\cdot\nabla b = 0 \tag{2}$$

$$\frac{1}{\rho_0}\partial_z p = b \tag{3}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{4}$$

Here $\mathbf{u}_h = (u, v)$ is the horizontal velocity and $\mathbf{u} = (u, v, w)$ is the full velocity; ∇_h is the horizontal part of the gradient operator; $f_0 + \beta y$ is the Coriolis parameter linearized about a reference latitude; ρ_0 is a constant reference density; p is pressure; $b = -g\delta\rho/\rho_0$ is buoyancy. We allow topography and an active free surface, and we impose wind forcing and bottom friction via linear Ekman layers[1] using the following boundary conditions

$$w - \mathbf{u}\cdot\nabla_h\eta_b - d_E\omega = 0 \quad \text{at}\, z = \eta_b \tag{5}$$

$$w - \partial_t\eta_t - \mathbf{u}\cdot\nabla_h\eta_t - \frac{\text{curl}[\boldsymbol{\tau}]}{f_0} = 0 \quad \text{at}\, z = H + \eta_t. \tag{6}$$

Here $\boldsymbol{\tau} = (\tau^x, \tau^y)$ is a two component vector denoting the wind stress applied at the surface, η_b is the height of topography, and η_t is the height of dynamic surface deformation above the mean depth H; ω is the vertical component of relative vorticity, and d_E is the Ekman layer depth.

Since we are allowing a free surface, it is convenient to account explicitly for its effect on the barotropic pressure. Integrating (3) from z to $H+\eta_t$ and assuming the atmospheric pressure at the surface to be constant (which we set to zero without loss of generality) we arrive at

$$-\frac{1}{\rho_0}p = -g(H + \eta_t - z) + \int_z^{H+\eta_t} b\,\mathrm{d}z'. \tag{7}$$

The momentum equations make use of the horizontal gradient of pressure, which is

$$-\frac{1}{\rho_0}\nabla_h p = -g\nabla_h\eta_t + \int_z^{H+\eta_t} \nabla_h b\,\mathrm{d}z' + (\nabla_h\eta_t)b(z = H + \eta_t). \tag{8}$$

In anticipation of the relative smallness of the surface height deformation, we linearize (8) as follows

$$-\frac{1}{\rho_0}\nabla_h p = -g\nabla_h\eta_t + \int_z^{H} \nabla_h b\,\mathrm{d}z'. \tag{9}$$

Substituting this into the horizontal momentum equation yields

$$\partial_t\mathbf{u}_h + \mathbf{u}\cdot\nabla\mathbf{u}_h + (f_0 + \beta y)\hat{\mathbf{z}}\times\mathbf{u}_h + g\nabla_h\eta_t - \int_z^{H} \nabla_h b\,\mathrm{d}z' = 0. \tag{10}$$

We nondimensionalize the equations using the eddy velocity scale[2] U, a generic horizontal length scale L, the horizontal advective time scale L/U and a generic buoyancy scale N^2H. We also scale the vertical velocity so that $w\sim HU/L$. At the risk of confusion, we make no notational distinction between dimensional and nondimensional quantities; past this point all our equations are nondimensional unless the surrounding text makes clear otherwise. The nondimensional equations and boundary conditions are

$$\partial_t\mathbf{u}_h + \mathbf{u}\cdot\nabla\mathbf{u}_h + \left(Ro^{-1} + \left(\frac{L}{L_\beta}\right)^2 y\right)\hat{\mathbf{z}}\times\mathbf{u}_h + Fr_e^{-2}A_{\eta,t}\nabla_h\eta_t - Fr_i^{-2}\int_z^1 \nabla_h b\,\mathrm{d}z' = 0 \tag{11}$$

$$\partial_t b + \mathbf{u}\cdot\nabla b = 0 \tag{12}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{13}$$

and

$$w - A_{\eta,b}\mathbf{u}\cdot\nabla_h\eta_b - E^{1/2}\omega = 0 \quad \text{at}\, z = 0 \tag{14}$$

$$w - A_{\eta,t}(\partial_t\eta_t + \mathbf{u}\cdot\nabla_h\eta_t) - A_w\text{curl}[\boldsymbol{\tau}] = 0 \quad \text{at}\, z = 1. \tag{15}$$

[1] The analysis could equally well be carried out using quadratic friction.
[2] Note that we do not use an externally imposed velocity scale; this is in contrast to common practice in studies of quasi-geostrophic turbulence which often use the velocity scale afforded by an imposed background shear.

The surface boundary conditions have been linearized to apply at $z=0$, 1. The Rhines scale is $L_\beta = \sqrt{U/\beta}$, and the nondimensional numbers are

$$Ro = \frac{U}{f_0 L}, \quad Fr_e = \frac{U}{\sqrt{gH}}, \quad Fr_i = \frac{U}{NH}, \quad E^{1/2} = \frac{d_E}{H}, \quad A_{\eta,t} = \frac{\eta_t^*}{H}, \quad A_{\eta,b} = \frac{\eta_b^*}{H}, \quad A_w = \frac{|\boldsymbol{\tau}|}{UHf_0}. \tag{16}$$

We add extra space and time coordinates so that T is the slow time variable, t is fast, X, Y are large, and x, y are small; $|\tau|$, η_t^*, and η_b^* denote the characteristic dimensions of the applied wind stress, free surface deviations, and topography, respectively. The slow time scale is advective on the large spatial scale, i.e. $T^* = L_{X,Y}/U$. This makes the time scale separation equal to the space scale separation. The addition of extra independent variables yields

$$A_h(\partial_T \mathbf{u} + \overline{\nabla}_h \cdot (\mathbf{uu})) + \partial_t \mathbf{u} + \nabla_h \cdot (\mathbf{uu}) + \partial_z(w\mathbf{u}) + (Ro^{-1} + A_h A_\beta^2 Y)\hat{\boldsymbol{z}} \times \mathbf{u}$$
$$+ (RoA_hA_e)^{-2} A_{\eta,t}(\nabla_h + A_h\overline{\nabla}_h)\eta_t - Fr_i^{-2}\int_z^1 (\nabla_h + A_h\overline{\nabla}_h) b \,\mathrm{d}z' = 0 \tag{17}$$

$$A_h(\partial_T b + \overline{\nabla}_h \cdot (\mathbf{u}b)) + \partial_t b + \nabla_h \cdot (\mathbf{u}b) + \partial_z(wb) = 0 \tag{18}$$

$$A_h \overline{\nabla}_h \cdot \mathbf{u} + \nabla_h \cdot \mathbf{u} + \partial_z w = 0 \tag{19}$$

$$w - A_{\eta,b}(A_h \mathbf{u} \cdot \overline{\nabla}_h \eta_b + \mathbf{u} \cdot \nabla_h \eta_b) - E\hat{\boldsymbol{z}} \cdot (\nabla \times + A_h \overline{\nabla}_h \times)\mathbf{u} = 0 \quad \text{at}\, z = 0 \tag{20}$$

$$w - A_h A_{\eta,t}(\partial_T \eta_t + \mathbf{u} \cdot \overline{\nabla}_h \eta_t) - A_{\eta,t}(\partial_t \eta_t + \mathbf{u} \cdot \nabla_h \eta_t) - A_w \hat{\boldsymbol{z}} \cdot (\nabla \times + A_h \overline{\nabla}_h \times)\boldsymbol{\tau} = 0 \ \text{at}\, z = 1 + A_{\eta,t}\eta_t. \tag{21}$$

Here $\overline{\nabla}_h = (\partial_X, \partial_Y)$ is the horizontal gradient acting on the large scale variables. The new nondimensional parameters are

$$A_h = \frac{L_{x,y}}{L_{X,Y}}, \quad A_\beta = \frac{L_{X,Y}}{L_\beta}, \quad A_e = \frac{L_{X,Y}}{L_e} \tag{22}$$

where $L_e = \sqrt{gH}/f_0$ is the external deformation radius; note $Fr_e = RoA_hA_e$.

We set the small scale equal to the baroclinic deformation radius $L_{x,y} = L_d = NH/f_0$. The separation between $L_{X,Y}$ and L_d defines a small asymptotic parameter ϵ, which is related to the other nondimensional parameters by the distinguished limit

$$A_h \equiv \epsilon \ll 1, \quad Fr_i \sim \epsilon, \quad E \sim \epsilon^2, \quad Ro \sim \epsilon, \quad A_{\eta,t} \sim \epsilon^2, \quad A_{\eta,b} \sim \epsilon^2, \quad A_w \sim \epsilon^2, \quad A_\beta \sim \mathcal{O}(1), \quad A_e \sim \mathcal{O}(1). \tag{23}$$

The distinguished limit sets the large scale $L_{X,Y}$ comparable to both the Rhines scale and the external deformation radius, but this is merely a convenience which allows us to investigate the effects of β and a free surface simultaneously. The ratios of the large scale to the Rhines scale (A_β) and to the external deformation radius (A_e) should be considered free parameters which are only constrained to be order one or less. This is made clear by the fact that the essentials of the following analysis are unchanged on an f-plane with a rigid lid, by setting either or both of A_β and A_e to zero (more precisely, one may take $A_e = \sqrt{A_{\eta,t}}/\epsilon$ and then let $A_{\eta,t}$ become smaller than ϵ^2).

The main requirement of the distinguished limit is that the Rhines scale L_β be greater than L_d; for this reason our analysis does not apply in the tropics where L_d exceeds the Rhines scale. The large scale $L_{X,Y}$ is also required to be smaller than the planetary scale, because the gradient of the Coriolis parameter on the large scale is not order one as it would be, for example, in planetary geostrophy (PG). The large scale is thus constrained by the distinguished limit, but is not as yet explicitly tied to any external parameters. In the following we will derive two equation sets; in the first the large scale is implicity defined to be the scale on which isopycnal variations are order one, and in the second it is the scale where kinetic energy is predominantly barotropic.

Having defined extra scales and introduced corresponding independent variables to the equations, we proceed to formulate mean (large scale) and eddy (small scale) equations as follows. We first define a formal multiscale average

$$\overline{\mathbf{u}}(X,Y,z,T)=\lim_{L\to\infty,t'\to\infty}\frac{1}{4t'L^2}\int_0^{t'}\int_{-L}^{L}\int_{-L}^{L}\mathbf{u}(X,x,Y,y,z,T,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \tag{24}$$

This allows the formal separation of all dependent variables into mean and eddy components, denoted by an overbar $\overline{(\cdot)}$ and a prime $(\cdot)'$ respectively. This formal average has the same properties as a Reynolds average, i.e. $\overline{\overline{A}}=\overline{A}$ and $\overline{AB}=\overline{A}\,\overline{B}+\overline{A'B'}$. We apply it to the multiscale equations (17)–(21) to arrive at equations for the mean variables.

$$\begin{aligned}&\epsilon(\partial_T\overline{\mathbf{u}}+\overline{\nabla}_h\cdot(\overline{\mathbf{u}}\,\overline{\mathbf{u}}))+\partial_z(\overline{w}\,\overline{\mathbf{u}})+\epsilon^{-1}(1+A_\beta^2\epsilon^2Y)\hat{\boldsymbol{z}}\times\overline{\mathbf{u}}\\&\quad+\epsilon^{-1}A_e^{-2}\overline{\nabla}_h\overline{\eta}_t-\epsilon^{-1}\int_z^1\overline{\nabla}_h\overline{b}\,\mathrm{d}z'=-\overline{\nabla}_h\cdot(\overline{\mathbf{u}'\mathbf{u}'})-\partial_z(\overline{w'\mathbf{u}'})\end{aligned} \tag{25}$$

$$\epsilon\overline{\nabla}_h\cdot\overline{\mathbf{u}}+\partial_z\overline{w}=0 \tag{26}$$

$$\epsilon(\partial_T\overline{b}+\overline{\nabla}_h\cdot(\overline{\mathbf{u}}\,\overline{b}))+\partial_z(\overline{w}\,\overline{b})=-\overline{\nabla}_h\cdot(\overline{\mathbf{u}'b'})-\partial_z(\overline{w'b'}) \tag{27}$$

$$\overline{w}-\epsilon^3(\overline{\mathbf{u}}\cdot\overline{\nabla}_h\overline{\eta}_b+\hat{\boldsymbol{z}}\cdot\overline{\nabla}_h\times\overline{\mathbf{u}})=\epsilon^2(\epsilon\overline{\mathbf{u}'\cdot\overline{\nabla}_h\eta_b'}+\overline{\mathbf{u}'\cdot\nabla_h\eta_b'})\quad\text{at}\,z=0 \tag{28}$$

$$\overline{w}-\epsilon^3(\partial_T\overline{\eta}_t+\overline{\mathbf{u}}\cdot\overline{\nabla}_h\overline{\eta}_t)=\epsilon^2\overline{\mathbf{u}'\cdot\nabla_h\eta_t'}+\epsilon^3(\overline{\mathbf{u}'\cdot\overline{\nabla}_h\eta_t'}+\hat{\boldsymbol{z}}\cdot\overline{\nabla}_h\times\overline{\boldsymbol{\tau}})\quad\text{at}\,z=1. \tag{29}$$

This derivation of the mean equations may be thought of as the application of a necessary solvability condition for the small-scale fast dynamics, since it requires $\overline{\partial_\tau\mathbf{u}'}=0$, and likewise for the averages of other small-scale derivatives.

We subtract the mean equations from the multiscale equations to arrive at the following equations for the eddies

$$\begin{aligned}&\epsilon(\partial_T\mathbf{u}'+\overline{\nabla}_h\cdot(\mathbf{u}\mathbf{u})')+\partial_t\mathbf{u}'+\nabla_h\cdot(\mathbf{u}\mathbf{u})'+\partial_z(w\mathbf{u})'+(\epsilon^{-1}+A_\beta^2\epsilon Y)\hat{\boldsymbol{z}}\times\mathbf{u}'\\&\quad+\epsilon^{-2}A_e^{-2}(\nabla_h+\epsilon\overline{\nabla}_h)\eta_t'-\epsilon^{-2}\int_z^1(\nabla_h+\epsilon\overline{\nabla}_h)b'\,\mathrm{d}z'=0\end{aligned} \tag{30}$$

$$\epsilon\overline{\nabla}_h\cdot\mathbf{u}'+\nabla_h\cdot\mathbf{u}'+\partial_zw'=0 \tag{31}$$

$$\epsilon(\partial_Tb'+\overline{\nabla}_h\cdot(\mathbf{u}b)')+\partial_tb'+\nabla_h\cdot(\mathbf{u}b)'+\partial_z(wb)'=0 \tag{32}$$

$$w'-\epsilon^2(\epsilon(\mathbf{u}\cdot\overline{\nabla}_h\eta_b)'+(\mathbf{u}\cdot\nabla_h\eta_b)')-\epsilon^2\hat{\boldsymbol{z}}\cdot(\nabla\times+\epsilon\overline{\nabla}_h\times)\mathbf{u}'=0\quad\text{at}\,z=0 \tag{33}$$

$$w'-\epsilon^3(\partial_T\eta_t'+(\mathbf{u}\cdot\overline{\nabla}_h\eta_t)')-\epsilon^2(\partial_t\eta_t'+(\mathbf{u}\cdot\nabla_h\eta_t)')-\epsilon^2\hat{\boldsymbol{z}}\cdot(\nabla\times+\epsilon\overline{\nabla}_h\times)\boldsymbol{\tau}'=0\quad\text{at}\,z=1. \tag{34}$$

The resulting eddy-mean system of Eqs. (25)–(29) and (30)–(34) may now be reduced simultaneously using standard asymptotic methods, e.g. the expansion of all dependent variables in asymptotic power series, etc.

3. The strong synoptic-scale isopycnal gradient limit

At leading order, the large scale equations are

$$\hat{\boldsymbol{z}} \times \overline{\mathbf{u}}_0 + A_e^{-2}\overline{\nabla}_h \overline{\eta}_{t,0} - \int_z^1 \overline{\nabla}_h \overline{b}_0 \,\mathrm{d}z' = 0 \tag{35}$$

$$\int_0^1 [\partial_T \overline{\omega}_0 + \overline{\mathbf{u}}_0 \cdot \overline{\nabla}_h \overline{\omega}_0 + A_\beta^2 \overline{v}_0]\,\mathrm{d}z - \partial_T \overline{\eta}_{t,0} + \overline{\omega}_0|_{z=0} = \mathrm{curl}[\overline{\boldsymbol{\tau}}] - \int_0^1 [\mathrm{curl}[\overline{\nabla}_h \cdot \overline{\mathbf{u}_0'\mathbf{u}_0'}]]\,\mathrm{d}z \tag{36}$$

$$\partial_T \overline{b}_0 + \overline{\mathbf{u}}_0 \cdot \overline{\nabla}_h \overline{b}_0 = 0. \tag{37}$$

Here and below subscripts denote asymptotic order, and ω denotes the horizontal component of relative vorticity, i.e. $\overline{\omega}_0 = \mathrm{curl}[\overline{\mathbf{u}}_0]$.

These equations have been called ‘Large Amplitude Geostrophic’ by Benilov (1993, 1994) because they allow order-one isopycnal deviations and ‘Frontal Geostrophic’ by (Zeitlin, 2008) due to their analogy with layered FG dynamics Cushman-Roisin (1986). We prefer the terminology of Benilov (1993) because the FG equations of Cushman-Roisin (1986) are valid to a higher order than the LAG equations, and contain cubic nonlinearities making them fundamentally distinct from LAG despite their similarities. It is worth noting that the LAG equations are derivable as a long-wave limit of the QG equations (Benilov et al., 1998), which implies that the QG approximation is consistent with large amplitude isopycnal deviations over long horizontal scales.

The large scale $L_{X,Y}$ in this system is implicitly equal to the scale on which isopycnal variation is order one, a situation common in, for example, the core of the Antarctic Circumpolar Current (ACC), arguably the region where mesoscale eddies exert the largest influence on the general circulation of the ocean. Note also that in the ACC the order one meridional variation of the isopycnals occurs on scales smaller than the planetary scale, yet much larger than the internal deformation radius.

Because the single-scale LAG equations are valid at scales just above those of baroclinic QG dynamics, their baroclinic instability to a mean geostrophic shear exhibits an ultraviolet catastrophe where the growth rate is unbounded at small scales (Benilov, 1993); the same is true for the PG equations (de Verdière, 1986). The asymptotics predict that the primary effect of deformation-scale dynamics on the synoptic dynamics is through the curl of a depth-averaged horizontal momentum flux. If this small-scale interaction term is sufficiently dissipative it might quell the ultraviolet catastrophe.

The leading-order eddy dynamics are quasigeostrophic dynamics on an f-plane with the influence of a horizontally uniform and time-independent background flow (supplied by the mean equations)

$$\hat{\boldsymbol{z}} \times \mathbf{u}_0' + A_e^{-2}\nabla_h \eta_{t,1}' - \int_z^1 \nabla_h b_1' \,\mathrm{d}z' = 0 \tag{38}$$

$$\partial_t q' + (\overline{\mathbf{u}}_0 + \mathbf{u}_0') \cdot \nabla_h q' + \mathbf{u}_0' \cdot \partial_z \left(\frac{\overline{\nabla}_h \overline{b}_0}{\partial_z \overline{b}_0} \right) = 0 \tag{39}$$

$$q' = \omega_0' + \partial_z \left(\frac{b_1'}{\partial_z \overline{b}_0} \right) \tag{40}$$

$$\partial_t b_1' + (\overline{\mathbf{u}}_0 + \mathbf{u}_0') \cdot \nabla_h b_1' + \mathbf{u}_0' \cdot \overline{\nabla}_h \overline{b}_0 = 0 \quad \text{at}\, z = 0, 1. \tag{41}$$

The system (35)–(41) includes two-way coupling between LAG and QG: large-scale baroclinic shear provided by LAG generates small-scale baroclinic instability in QG, and the resulting momentum flux in turn affects the LAG dynamics. However, the large-scale buoyancy equation does not include an eddy flux term, thus eddies cannot directly alter the stratification. Although only the depth-integrated component of the momentum flux appears at this order in the asymptotics, one could add next order terms to the system, for example by including the heat flux generated by the QG eddies in the LAG buoyancy equation, or by including Ekman layer dissipation in the QG equations. This system is potentially useful for the study of the interaction of large-scale fronts with deformation-scale eddies. A similar set of equations wherein the small scale dynamics are geostrophic but not hydrostatic has been previously derived by K. Julien and G. Vasil (personal communication).

4. The weak synoptic-scale isopycnal gradient limit

The above equations are valid when the scale of isopycnal variation is smaller than the planetary scale, for example in the vicinity of large-scale fronts. Away from such fronts the isopycnal variation becomes order one only on the planetary scale, which by assumption is much larger than $L_{X,Y}$. In these regions the dynamics on the scale of isopycnal variation are described by PG, and the small-scale coupling is investigated using MSA by Pedlosky (1984) and Grooms et al. (2011). However, in those same regions it is possible to examine dynamics at a scale intermediate between the deformation radius and the planetary scale; at high latitudes for example there is a very great disparity in scale between the planetary scale and the deformation radius.

Because our large scale $L_{X,Y}$ is not explicitly tied to any external parameters, we can proceed in the same framework to examine the dynamics with small isopycnal variation at scales between the planetary scale and the deformation radius. Two properties of the LAG equations allow us to proceed: they do not evolve the horizontal mean of the stratification, and if supplied with a horizontally uniform initial buoyancy they will not generate horizontal buoyancy variations. That is to say, $b_0 = b_0(z)$ is an exact, linearly stable solution of the LAG equations. We may thus consistently proceed to next order in the asymptotics of the large scale buoyacy equation by setting $\overline{b}_0 = \overline{b}_0(z)$. In order to include the effects of a gyre-scale buoyancy gradient whose slow evolution we do not specify, we modify the ansatz to $\overline{b}_0 = \overline{b}_0(z, \epsilon\delta X, \epsilon\delta Y)$ where δ is a parameter of at most order one which controls the strength of the externally imposed planetary scale isopycnal tilt.

Setting $\overline{b}_0 = \overline{b}_0(z, \epsilon\delta X, \epsilon\delta Y)$, the eddy-mean system becomes

$$\hat{\boldsymbol{z}} \times \overline{\mathbf{u}}_0 + A_e^{-2}\overline{\nabla}_h\overline{\eta}_{t,0} = 0, \quad \overline{b}_0 = \overline{b}_0(z, \epsilon\delta X, \epsilon\delta Y) \tag{42}$$

$$\partial_T\overline{q} + \overline{\mathbf{u}}_0 \cdot \overline{\nabla}_h\overline{q} - \mathrm{curl}[\overline{\boldsymbol{\tau}}] + \overline{\omega}_0 = -\mathrm{curl}[\overline{\nabla}_h \cdot \int_0^1 \overline{\mathbf{u}_0'\mathbf{u}_0'}\,\mathrm{d}z] \tag{43}$$

$$\partial_T\overline{b}_1 + \overline{\mathbf{u}}_0 \cdot \overline{\nabla}_h\overline{b}_1 + \delta\overline{\mathbf{u}}_0 \cdot \overline{\nabla}_h\overline{b}_0 = -\overline{\nabla}_h \cdot (\overline{\mathbf{u}_0'b_1'}) \tag{44}$$

$$\overline{q} = (\overline{\omega}_0 - \overline{\eta}_{t,0} + \overline{\eta}_b + A_\beta^2 Y) \tag{45}$$

$$\hat{\boldsymbol{z}} \times \mathbf{u}_0' + A_e^{-2}\nabla_h\eta_{t,1}' - \int_z^1 \nabla_h b_1'\,\mathrm{d}z' = 0 \tag{46}$$

$$\partial_t q' + (\overline{\mathbf{u}}_0 + \mathbf{u}_0') \cdot \nabla_h q' = 0 \tag{47}$$

$$q' = \omega_0' + \partial_z\left(\frac{b_1'}{\partial_z\overline{b}_0}\right) \tag{48}$$

$$\partial_t b_1' + (\overline{\mathbf{u}}_0 + \mathbf{u}_0') \cdot \nabla_h b_1' = 0 \quad \text{at}\, z = 0, 1 \tag{49}$$

To derive the mean equations we have used $\overline{w_1'b_1'} = 0$, which is a necessary solvability condition on the eddy dynamics: lacking friction, the average rate of generation of eddy kinetic energy (equal to $\overline{w_1'b_1'}$) must be zero or the eddy kinetic energy will grow secularly on the fast time scale.

Although these equations are closed, they are missing a key ingredient: the above equations do not account for changes in eddy energy due to interactions with the mean flow, because there is no baroclinic instability term in the small-scale equations. On the other hand, the large-scale buoyancy equation now includes an eddy flux term, thus eddies can directly affect the stratification. Energy transfer to and from the large scales in the above system occurs only on the slow time scale, whereas the eddy equations describe evolution only on the fast time scale; if one can account for the slow-time evolution of the eddy energy, it will be possible to close the energy budget.

4.1. Slow-time evolution of the eddy energy

In the appendix we derive an equation for the slow-time, large-scale evolution of eddy energy as a necessary solvability condition on the next-order eddy dynamics. The resulting energy evolution equation is

$$\partial_T\langle E\rangle + \overline{\mathbf{u}}_0\cdot\overline{\nabla}_h\langle E\rangle + \overline{\nabla}_h\cdot\langle\mathbf{u}_0'E\rangle - \overline{\nabla}_h\times\langle p_1'(\partial_t\mathbf{u}_0' + \overline{\mathbf{u}}_0\cdot\nabla_h\mathbf{u}_0')\rangle = -\langle\mathbf{u}_0'\cdot(\mathbf{u}_0'\cdot\overline{\nabla}_h\overline{\mathbf{u}}_0)\rangle$$
$$- \left\langle \frac{\overline{\mathbf{u}_0'b_1'}\cdot(\overline{\nabla}_h\bar{b}_1 + \delta\overline{\overline{\nabla}}_h\bar{b}_0)}{\partial_z\bar{b}_0}\right\rangle + \overline{\mathbf{u}_0'|_{z=1}\cdot\boldsymbol{\tau}} - \overline{|\mathbf{u}_0'|^2|_{z=0}} + \overline{p_1|_{z=0}'(\overline{\mathbf{u}}_0\cdot\nabla_h\eta_b')}. \tag{50}$$

where $E=(1/2)(|\mathbf{u}_0'|^2+((b_1')^2/(\partial_z\bar{b}_0)))$ and $\langle\cdot\rangle$ denotes averaging over depth in addition to the multiscale average $\overline{(\cdot)}$. The above equation guarantees energetic consistency for the eddy-mean system, so that energy gained or lost by the mean flow due to eddy forcing is reflected in a corresponding loss or gain by the eddies. Application of the same methods to homogeneous isotropic turbulence yields a similar equation for the slow evolution of small scale energy (McLaughlin et al., 1985).

The large scale dynamics in this system are simply the large scale dynamics which occur in regimes of QG turbulence with an extended inverse cascade: the large scale vorticity is primarily barotropic, and the large scale buoyancy field is passively advected by the barotropic flow. The inverse cascade is accomplished by the eddy momentum flux, which provides small-scale forcing to the large-scale barotropic vorticity equation. The generation and forward cascade of potential energy[3] is accomplished by passive advection of large scale buoyancy by the barotropic flow, being ultimately absorbed by the eddy buoyancy flux divergence. The eddy dynamics are highly energetic; so much so, in fact, that their nonlinear self interaction occurs on a timescale faster than the timescale of baroclinic instability. Energy exchange with the mean flow occurs slowly in comparison with advection on the small scales, and this effect is included through the eddy energy equation (50).

5. Discussion and conclusions

We have used MSA with a single distinguished limit to derive two systems of equations describing the interaction of synoptic scales and mesoscales in the oceans. In both systems the small scale is comparable to the deformation radius, and bottom friction acts at leading order on the large scale velocity. The distinguished limit requires the synoptic β-scale and external deformation radius to be larger than the internal deformation radius, so the analysis is not applicable to the tropics. The synoptic scale (our relative large scale) in both systems is smaller than the planetary scale, while the large scale in Pedlosky (1984) and Grooms et al. (2011) is equal to the planetary scale. The first system, (35)–(41), describes the interaction of large scale 'Large Amplitude Geostrophic' (LAG) dynamics with small scale eddies (QG). The dynamics are coupled by an eddy momentum flux in the LAG equations and by baroclinic instability of the small scale dynamics to the large-scale shear. This system is applicable to regions where the scale of order one isopycnal variation is larger than the deformation radius but smaller than the planetary scale, for example in the vicinity of moderately large scale baroclinic fronts or near the boundaries of wind driven gyres.

Mathematically, the equations are in need of regularization, principally because the QG equations include baroclinic instability but lack dissipation, but also because of the catastrophic baroclinic instability present in the LAG equations (Benilov, 1993). This situation is similar to the difficulties encountered with the PG equations, which require the addition of frictional and dissipative terms (e.g. de Verdière, 1986, 1988; Samelson and Vallis, 1997; Samelson et al., 1998). Our asymptotic analysis shows that the small-scale contribution to the large-scale dynamics in this regime is dominantly through the divergence of the horizontal momentum flux; indeed it is possible that this term may be

[3] Quasigeostrophic available potential energy, to be precise, which is equal to half the square of the buoyancy variance.

sufficient to regularize the instability of LAG if it is sufficiently dissipative though a simple demonstration is lacking due to the nonlinearities involved. The inclusion of next-order asymptotic terms like Ekman friction in the QG equations and eddy buoyancy flux in the LAG equations would improve the ability of this system to reliably model the interactions between dynamics at these scales.

The second system, (42)–(50), describes the interaction of large scale barotropic QG dynamics with deformation-scale baroclinic QG dynamics in quasigeostrophic turbulence with an extended inverse cascade. The dynamics are coupled by eddy momentum and buoyancy fluxes in the synoptic equations and by an equation for the slow evolution of eddy energy (50). This system is applicable in regions where isopycnal variation remains small on the synoptic scale.

The second system can be thought of as a model for the classic QG baroclinic turbulence cycle, with forcing by a mean shear and dissipation by weak drag (e.g. Salmon, 1980, 1998; Held and Larichev, 1996; Larichev and Held, 1995, and many others). The large scale flow is primarily barotropic. Large scale potential energy is generated by the interaction of the barotropic flow with the background 'planetary' buoyancy gradient, and cascades downscale as the buoyancy is passively advected by the barotropic flow. Near the deformation scale potential energy is converted to kinetic energy through baroclinic instability, which undergoes an upscale cascade and barotropizes before being dissipated at the large scales by bottom friction. In our second system, the large scale buoyancy is passively advected by the barotropic flow, and is generated by the background 'planetary' buoyancy gradient. As buoyancy variance (QG available potential energy) cascades downscale, it is absorbed by the eddies through the divergence of a buoyancy flux. The slow, large scale eddy energy equation (50) captures this exchange of energy with the mean flow. The QG eddy dynamics generate a momentum flux that provides a small scale forcing term in the large scale barotropic vorticity equation, resulting in a cascade of energy to larger scales, where it is ultimately absorbed by the bottom friction, or some other process.

A potential weakness of this system is that on the fast time scale the eddy dynamics only feel the barotropic mean flow; since these fast dynamics do not know about the direction of the large scale buoyancy gradient, it is unreasonable to suspect that they will produce a buoyancy flux which is down the mean gradient on average. This is consistent with the results of Nadiga (2008) who found that the eddy PV flux in a quasigeostrophic system was only weakly correlated with the mean PV gradient, being weakly downgradient only in an average sense. Our multiple scale asymptotic analysis provides a description only of the direct interaction between motions at the deformation radius and motions at the synoptic scale. In QG turbulence the interaction between the deformation scale and the largest scales is not primarily direct, but rather goes through the intermediate scales which are ignored by the asymptotics.

In both systems the eddies drive the mean momentum through a depth-integrated momentum flux of the form

$$\mathrm{curl}[\overline{\nabla}_h \cdot \int_0^1 \overline{\mathbf{u}'\mathbf{u}'}\,\mathrm{d}z] = \int_0^1 [\partial_X\partial_Y(\overline{(v')^2} - \overline{(u')^2}) + (\partial_X^2 - \partial_Y^2)\overline{u'v'}]\,\mathrm{d}z. \tag{51}$$

This may be written in terms of the horizontal Fourier transform of the eddy streamfunction $\hat{\psi}'_k(z, \tau)$

$$\mathrm{curl}[\overline{\nabla}_h \cdot \int_0^1 \overline{\mathbf{u}'\mathbf{u}'}\,\mathrm{d}z] = \int_0^1 [\partial_X\partial_Y(\overline{(k_y^2 - k_x^2)|\hat{\psi}_k|^2}^k) - (\partial_X^2 - \partial_Y^2)\overline{k_y k_x|\hat{\psi}_k|^2}^k]\,\mathrm{d}z, \tag{52}$$

where $\overline{(\,\cdot\,)}^k$ denotes integration over horizontal wavenumbers in addition to a time average. The time-averaged 2D eddy kinetic energy spectrum at any depth is $\epsilon\int_0^{\epsilon^{-1}}(k_x^2 + k_y^2)|\hat{\psi}_k|^2\,\mathrm{d}\tau$; thus, (an)isotropy in the time averaged eddy kinetic energy spectrum corresponds to (an)isotropy in the time average of $|\hat{\psi}_k|^2$. If the time and depth averaged eddy energy spectrum is isotropic, then by symmetry

$$\int_0^1 (\overline{(k_y^2 - k_x^2)|\hat{\psi}_k|^2}^k)\,\mathrm{d}z = \int_0^1 \overline{k_y k_x|\hat{\psi}_k|^2}^k\,\mathrm{d}z = 0, \tag{53}$$

which implies that $\overline{(u')^2} = \overline{(v')^2}$ and $\overline{u'v'} = 0$; in such a case the eddy terms drop out of the synoptic scale vorticity equation in both systems. Our analysis thus links nontrivial eddy momentum forcing to anisotropy in the time and depth averaged eddy energy spectrum. In the first system, (35)–(41), anisotropy is generated by baroclinic interaction terms in the eddy equations and nontrivial eddy forcing is expected. In the second system, (42)–(50), eddy anisotropy is not expected at leading order, except as random fluctuations around an isotropic ensemble mean.

With sufficient regularization, the first model may serve as a useful testbed for the "superparameterization" of mesoscale eddies, and for the investigation of the interaction of strong, synoptic-scale fronts with smaller-scale QG eddies. We intend to use the second model to investigate the question of eddy locality (cf. Venaille et al. (2011)) by diagnosing local and nonlocal terms in the eddy energy budget of simulations, to see how well local baroclinic eddy energy generation is balanced by local eddy energy dissipation. The second model also suggests a stochastic approach: because large-scale interaction terms are absent from the leading order eddy equations in this model, the fluxes generated by the eddies are expected to have a significant random component, being only weakly correlated with the gradients of the mean variables. Stochastic models of QG dynamics are well developed (see DelSole, 2004 for a review); development of a parameterization based on stochastic modeling in conjunction with multiscale methods like superparameterization Majda (2012) and an eddy energy equation in the style of Large Eddy Simulation (LES) is a subject of further research.

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Appendix A. Details in the derivation of Eq. (50)

The derivation of Eq. (50) for the slow evolution of the eddy energy proceeds from the following set

$$\hat{\boldsymbol{z}} \times \mathbf{u}_0' = -\nabla_h p_1' \tag{A.1}$$

$$\partial_z p_1' = b_1' \tag{A.2}$$

$$\overline{\nabla}_h \cdot \mathbf{u}_0' + \nabla_h \cdot \mathbf{u}_1' + \partial_z w_1' = 0 \tag{A.3}$$

$$\overline{\nabla}_h \cdot \mathbf{u}_1' + \nabla_h \cdot \mathbf{u}_2' + \partial_z w_2' = 0 \tag{A.4}$$

$$\partial_t \mathbf{u}_0' + (\overline{\mathbf{u}}_0 + \mathbf{u}_0') \cdot \nabla_h \mathbf{u}_0' + \hat{\boldsymbol{z}} \times \mathbf{u}_1' = -\nabla_h p_2' - \overline{\nabla}_h p_1' \tag{A.5}$$

$$\begin{aligned} \partial_T \mathbf{u}_0' &+ \partial_t \mathbf{u}_1' + (\overline{\mathbf{u}}_0 + \mathbf{u}_0') \cdot \nabla_h \mathbf{u}_1' + \mathbf{u}_1 \cdot \nabla \mathbf{u}_0' + (\mathbf{u}_0 \cdot \overline{\nabla}_h \mathbf{u}_0)' + (w_1' \partial_z \mathbf{u}_0')' \\ &+ \hat{\boldsymbol{z}} \times \mathbf{u}_2' + A_\beta^2 Y \hat{\boldsymbol{z}} \times \mathbf{u}_0' = -\nabla_h p_3' - \overline{\nabla}_h p_2' \end{aligned} \tag{A.6}$$

$$\partial_t b_1' + (\overline{\mathbf{u}}_0 + \mathbf{u}_0') \cdot \nabla_h b_1' + w_1' \partial_z \overline{b}_0 = 0 \tag{A.7}$$

$$\partial_T b_1' + \partial_t b_2' + (\overline{\mathbf{u}}_0 + \mathbf{u}_0') \cdot \nabla_h b_2' + \mathbf{u}_1 \cdot \nabla_h \mathbf{u}_1' + \delta \mathbf{u}_0' \cdot \overline{\nabla}_h \overline{b}_0 + w_1' \partial_z \overline{b}_1 + w_2' \partial_z \overline{b}_0 + (w_1' \partial_z b_1')' = 0. \tag{A.8}$$

For brevity of notation, we have re-introduced the nondimensional pressure p. It is also useful to recall

$$\overline{w_1' b_1'} = \overline{w_1' (b_1')^2} = 0 \tag{A.9}$$

which can be derived by multiplying (A.7) by b_1' or $(b_1')^2$ and requiring the multiscale average of exact derivatives to vanish. The relevant boundary conditions are

$$w_1' = 0, \quad w_2' = (\mathbf{u}_0 \cdot \nabla_h \eta_b)' + \hat{\boldsymbol{z}} \cdot (\nabla \times \mathbf{u}_0') \quad \text{at } z = 0 \tag{A.10}$$

$$w_1' = 0, \quad w_2' = \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{\tau}' \quad \text{at } z = 1. \tag{A.11}$$

The derivation proceeds by taking the dot products of $\mathbf{u}'_1$ with (A.5) and $\mathbf{u}'_0$ with (A.6) and averaging over small scales, fast time, and the vertical coordinate. In this final step the averages of small scale and fast time derivatives are set to zero; this is effectively the application of a necessary solvability condition. The resulting equation describes the large scale evolution of eddy kinetic energy. The evolution of quasigeostrophic available potential energy is derived by multiplying (A.7) by b'_2 and (A.8) by b'_1, adding the results, dividing by $\partial_z\overline{b}_0$, and averaging over small scales, fast time, and the vertical coordinate. The net eddy energy equation is derived by summing the kinetic and potential energy equations.

The main difficulty in the otherwise straightforward derivation concerns the pressure terms; we therefore present that portion of the derivation here. Summing the contractions of (A.5) with $\mathbf{u}'_1$ and (A.6) with $\mathbf{u}'_0$ generates the following

$$\frac{1}{2}\partial_t|\mathbf{u}'_0|^2+\ldots+\mathbf{u}'_0\times\mathbf{u}'_2=-\mathbf{u}'_0\cdot\nabla_h p'_3-\mathbf{u}'_0\cdot\overline{\nabla}_h p'_2-\mathbf{u}'_1\cdot\nabla_h p'_2-\mathbf{u}'_1\cdot\overline{\nabla}_h p'_1. \tag{A.12}$$

Recalling that $\hat{\boldsymbol{z}}\times\mathbf{u}'_0=-\nabla_h p'_1$, we have $\mathbf{u}'_0\times\mathbf{u}'_2=\mathbf{u}'_2\cdot\nabla_h p'_1$. We insert this above and average over fast times and small scales. Integrating terms of the form $\mathbf{u}'\cdot\nabla_h p'$ by parts and making use of (A.3) and (A.4) results in

$$\frac{1}{2}\partial_t\overline{|\mathbf{u}'_0|^2}+\ldots=-\overline{\nabla}_h\cdot(\overline{p'_1\mathbf{u}'_1})-\overline{\nabla}_h\cdot(\overline{p'_2\mathbf{u}'_0})-\overline{p'_2\partial_z w'_1}-\overline{p'_1\partial_z w'_2}. \tag{A.13}$$

Again recalling that $\hat{\boldsymbol{z}}\times\mathbf{u}'_0=-\nabla_h p'_1$ we have $\overline{\nabla}_h\cdot(\overline{p'_1\mathbf{u}'_1}+\overline{p'_2\mathbf{u}'_0})=\overline{\nabla}_h\times(\overline{p'_1(\hat{\boldsymbol{z}}\times\mathbf{u}'_1+\nabla_h p'_2)})$. Eq. (A.5) allows further simplification to

$$\overline{\nabla}_h\cdot(\overline{p'_1\mathbf{u}'_1}+\overline{p'_2\mathbf{u}'_0})=-\overline{\nabla}_h\times(\overline{p'_1(\partial_\tau\mathbf{u}'_0+(\mathbf{u}'_0+\overline{\mathbf{u}}_0)\cdot\nabla_h\mathbf{u}'_0+\overline{\nabla}_h p'_1)}) \tag{A.14}$$

$$\overline{\nabla}_h\cdot(\overline{p'_1\mathbf{u}'_1}+\overline{p'_2\mathbf{u}'_0})=-\overline{\nabla}_h\times(\overline{p'_1(\partial_\tau\mathbf{u}'_0+\overline{\mathbf{u}}_0\cdot\nabla_h\mathbf{u}'_0)}). \tag{A.15}$$

This final simplification above makes use of the facts that $\overline{p'_1\mathbf{u}'_0\cdot\nabla_h\mathbf{u}'_0}=\overline{p'_1\nabla_h\cdot(\mathbf{u}'_0\mathbf{u}'_0)}=-\overline{\mathbf{u}'_0\times(\mathbf{u}'_0\mathbf{u}'_0)}=0$ and $\overline{\nabla}_h\times(p'_1\overline{\nabla}_h p'_1)=\overline{\nabla}_h\times\overline{\nabla}_h(p'^2_1)/2=0$. While the final result above is not equal to the pressure work, it is related to it, as shown by Pedlosky (1987).

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