Embeddings of the Heisenberg group and the Sparsest Cut problem

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(joint work with Assaf Naor)

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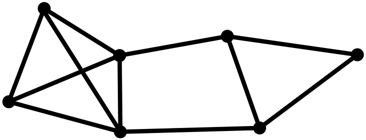

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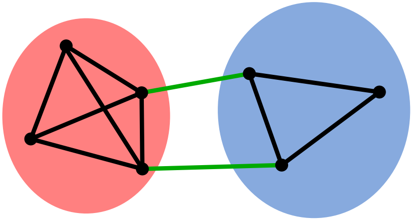
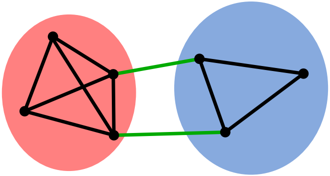


Sparsest Cut is a matrix problem



$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

If C is the adjacency matrix of G, then

$$\Phi(G) = \min_{S \subset V(G)} \frac{|E(S, S^c)|}{|S| \cdot |S^c|} = \min_{S \subset [n]} \frac{\sum_{i \in S, j \in S^c} C_{ij}}{\sum_{i \in S, j \in S^c} 1}$$

(where $[n] = \{1, \ldots, n\}$)

The Nonuniform Sparsest Cut problem

Problem

Let C (capacity) and D (demand) be symmetric $n \times n$ matrices with non-negative entries. Find:

$$\Phi(C, D) = \min_{S \subset [n]} \frac{\sum_{i \in S, j \in S^c} C_{ij}}{\sum_{i \in S, j \in S^c} D_{ij}}.$$

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This problem is NP-hard, but there are polynomial-time algorithms to approximate $\Phi(C, D)$ based on metric embeddings.

Relaxing the problem

A *cut metric* is a semimetric of the form

$$d_S(i,j) = |\mathbf{1}_S(i) - \mathbf{1}_S(j)| \qquad \text{where } S \subset X.$$

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and let $\mathcal{K} \supset \mathcal{C}$. The *relaxation* $\Phi_{\mathcal{K}}$ of Sparsest Cut is

$$\Phi_{\mathcal{K}}(C,D) = \min_{M \in \mathcal{K}} \frac{\sum_{i,j} C_{ij} M_{ij}}{\sum_{i,j} D_{ij} M_{ij}}.$$

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- Then $\Phi_{\mathcal{K}} \leq \Phi_{\mathcal{C}} = \Phi$.
- Is there a $\mathcal{K}$ such that $\Phi_{\mathcal{K}}$ is easy to compute and close to Φ?

Geometric relaxations

If $f : X \to Y$, let $d_f \in M_n$ be the induced distance function $d_f(i, j) = d(f(i), f(j))$.

- If $\mathcal{K}_1 = \{d_f \mid f : [n] \to L_1\}$, then $\Phi = \Phi_{\mathcal{K}_1}$ (Linial-London-Rabinovich)

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  Proof: Every n-point metric space embeds in L_1 with $\log n$ distortion (Bourgain)

The Goemans-Linial relaxation

Theorem (Goemans-Linial)

Let $\mathcal{N} = \{n \times n$ distance matrices with negative type$\}$. Then $\mathcal{K}_1 \subset \mathcal{N} \subset \mathcal{M}$, so

$$\frac{\Phi}{\log n} \lesssim \Phi_{\mathcal{M}} \leq \Phi_{\mathcal{N}} \leq \Phi.$$

Furthermore, $\Phi_{\mathcal{N}}$ can be computed in polynomial time.

The Goemans-Linial question

Define the *Goemans-Linial integrality gap* $\alpha(n) = \max \frac{\Phi(C,D)}{\Phi_{\mathcal{N}}(C,D)}$ where C, D are $n \times n$ matrices.

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- $\alpha(n) \gtrsim (\log \log n)^c$ (Khot-Vishnoi)
- $\alpha(n) \gtrsim (\log n)^{c'}$ (with $c' \approx 2^{-60}$) (Cheeger-Kleiner-Naor)

The main theorem

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Corollary (Naor-Y.)

$$\alpha(n) \gtrsim \sqrt{\log n}$$

Part 2: The Heisenberg group

Let $H^{2k+1} \subset M_{k+2}$ be the $(2k+1)$–dimensional nilpotent Lie group

$$H^{2k+1} = \left\{ \begin{pmatrix} 1 & x_1 & \dots & x_k & z \\ 0 & 1 & 0 & 0 & y_1 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & y_k \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x_i, y_i, z \in \mathbb{R} \right\}.$$

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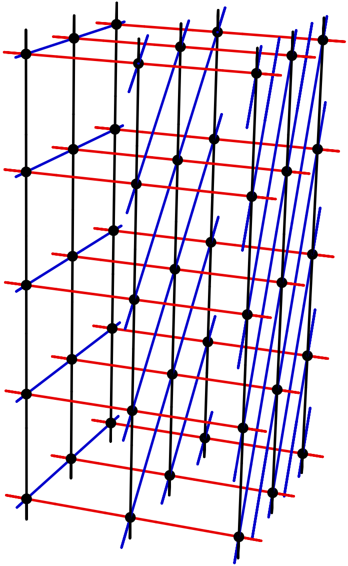
This contains a lattice

$$H^{\mathbb{Z}}_{2k+1} = \langle x_1, \dots, x_k, y_1, \dots, y_k, z \mid [x_i, y_i] = z, \text{ all other pairs commute} \rangle.$$

A lattice in H^3



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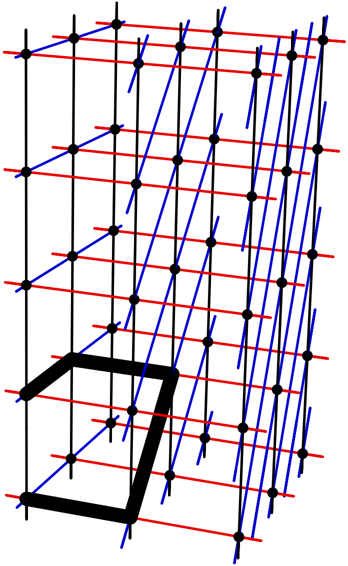

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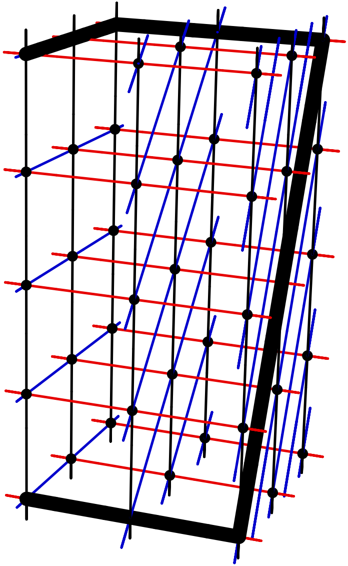

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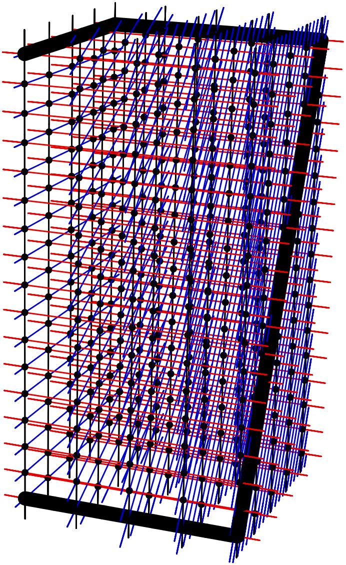

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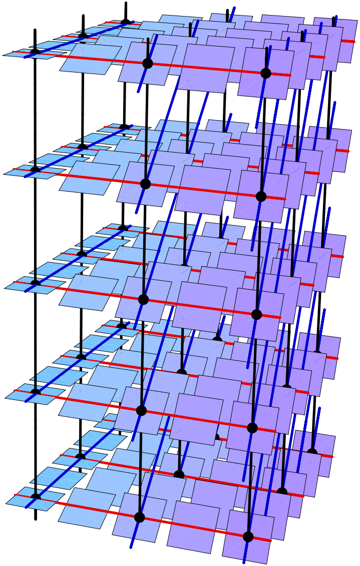
From Cayley graph to sub-riemannian metric



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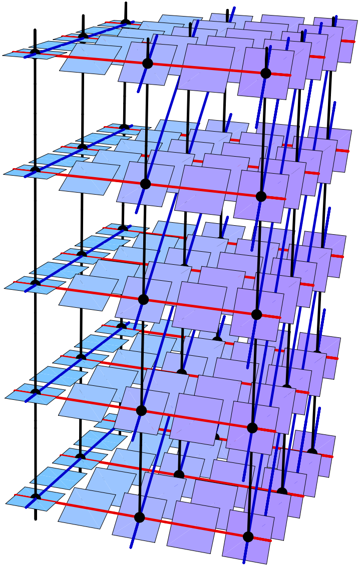

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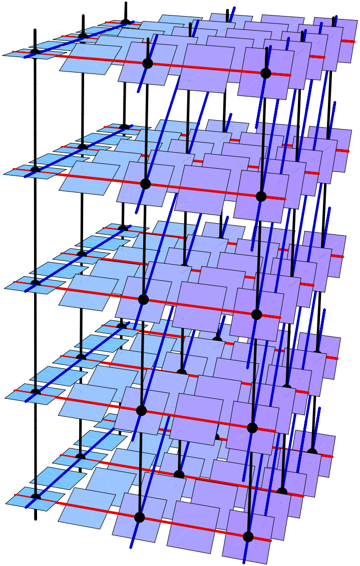

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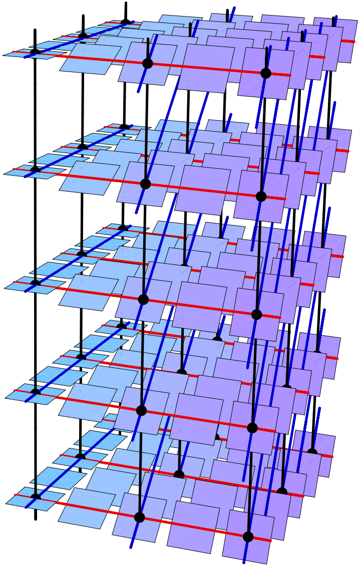

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- The z–axis has Hausdorff dimension 2

The Heisenberg group and the Goemans–Linial question

Theorem (Lee-Naor)

The sub-riemannian metric on the Heisenberg group is bilipschitz equivalent to a metric of negative type.

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Corollary (Cheeger-Kleiner)

There are finite subsets of H^{2k+1} that do not embed bilipschitzly in L_1. (i.e., counterexamples to the Goemans–Linial question)

Part 3: Nonembeddability of the Heisenberg group

Theorem (Pansu, Semmes)

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That is, on sufficiently small scales, f is close to a homomorphism. But any homomorphism sends z to 0 – so any Lipschitz map to $\mathbb{R}^N$ collapses the z direction.

H^{2k+1} does not embed in L_1

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Example

The map $f : [0,1] \to L_1([0,1])$

$$f(t) = \mathbf{1}_{[0,t]},$$

is an isometric embedding that cannot be approximated by a linear map.

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Proof

Let B be a ball in H^{2k+1}. Every L_1-metric on B is a linear combination of cut metrics:

Lemma

If $f : B \to L_1$, then there is a measure μ (the cut measure) on 2^B such that

$$d(f(x), f(y)) = \int d_S(x, y)\, d\mu(S).$$

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We can thus study f by studying cuts in H^{2k+1}.

Proof: H^{2k+1} does not embed in L_1

Open sets in H^{2k+1} have Hausdorff dimension $2k+2$ and any surface that separates two open sets has Hausdorff dimension at least $2k+1$, so we let area $= \mathcal{H}^{2k+1}$, vol $= \mathcal{H}^{2k+2}$.

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Lemma

If $B \subset H^{2k+1}$ is the unit ball and $f : B \to L_1$ is Lipschitz, then the cut measure μ is supported on sets S with $\mathrm{area}(\partial S) < \infty$ *and*

$$\int \mathrm{area}(\partial S)\, d\mu(S) \lesssim \mathrm{vol}(B)\,\mathrm{Lip}(f).$$

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If area $\partial S < \infty$, *then near almost every* $x \in \partial S$, ∂S *is close to a plane containing the z–axis (the tangent plane at* x.*)*

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- For almost every $x \in B$, there is a neighborhood B' of x such that most of the cuts are close to vertical on B'.

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- Therefore, $f|_{B'}$ is close to a map that is constant on vertical lines.
- So f is not a bilipschitz map.

Quantitative nonembeddability

Cheeger, Kleiner, and Naor quantified this result:

Theorem (Cheeger-Kleiner-Naor)

Let $B \subset H^3$ be the ball of radius 1. There is a $\delta > 0$ such that for any $\epsilon > 0$ and any 1–Lipschitz map $f : B \to L_1$, there is a ball B' of radius at least ϵ such that $f|_{B'}$ is $\asymp |\log \epsilon|^{-\delta}$–close to a map that is constant on vertical lines.

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Corollary

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$$\alpha(n) \gtrsim (\log n)^{\delta}.$$

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$$\alpha(n) \gtrsim (\log n)^{\delta}.$$

But δ is tiny – around 2^{-60}.

The main theorem

Theorem (Naor-Y.)

Let $k \geq 2$ and let $B \subset H^{2k+1}$ be the unit ball. Let $Z \in H^{2k+1}$ generate the z–axis. If $f : H^{2k+1} \to L_1$ is Lipschitz, then

$$\int_0^1 \left(\int_B \frac{\|f(x) - f(xZ^t)\|_1}{d(x, xZ^t)} \, dx \right)^2 \frac{dt}{t} \lesssim \mathrm{Lip}(f)^2.$$

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If f were bilipschitz, then this integral would be infinite, so

Corollary

B does not embed bilipschitzly in L_1.

The main theorem

Theorem (Naor-Y.)

Let $k \geq 2$ and let $B \subset H^{2k+1}$ be the unit ball. Let $Z \in H^{2k+1}$ generate the z–axis. If $f : H^{2k+1} \to L_1$ is Lipschitz, then

$$\int_0^1 \left(\int_B \frac{\|f(x) - f(xZ^t)\|_1}{d(x, xZ^t)} \, dx \right)^2 \, \frac{dt}{t} \lesssim \mathrm{Lip}(f)^2.$$

If f were bilipschitz, then this integral would be infinite, so

Corollary

B does not embed bilipschitzly in L_1.

And this gives sharp bounds on the scale of the distortion:

Corollary

The Goemans-Linial integrality gap $\alpha(n)$ is bounded by

$$\alpha(n) \gtrsim \sqrt{\log n}.$$

Reducing to surfaces

The sharp bound on Lipschitz embeddings follows from:

Theorem (Naor-Y.)

Let $k \geq 2$ *and let* $S \subset H^{2k+1}$ *be a set with* $\text{area}\,\partial S < \infty$. *Let*

$$S \triangle T = (S \setminus T) \cup (T \setminus S)$$

Then

$$\int_0^\infty \frac{\text{vol}(S \triangle SZ^t)^2}{t^2}\, dt \lesssim \text{area}(\partial S)^2.$$

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Theorem (David-Semmes)

A set $E \subset \mathbb{R}^k$ is uniformly rectifiable if and only if E has a corona decomposition. (Roughly, for all but a few balls B, the intersection $B \cap E$ is close to the graph of a Lipschitz function with small Lipschitz constant.)

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- Naor-Y.: Surfaces in H^{2k+1} are made of *uniformly rectifiable* pieces.

Decompositions in $\mathbb{R}^k$ and H^{2k+1}

Theorem (Y.)

If T is a mod-2 d–cycle in $\mathbb{R}^k$, $d < k$, it can be decomposed as a sum $T = \sum_i T_i$ such that $\operatorname{supp} T_i$ *is uniformly rectifiable and* $\sum_i \operatorname{mass} T_i \lesssim \operatorname{mass} T$.

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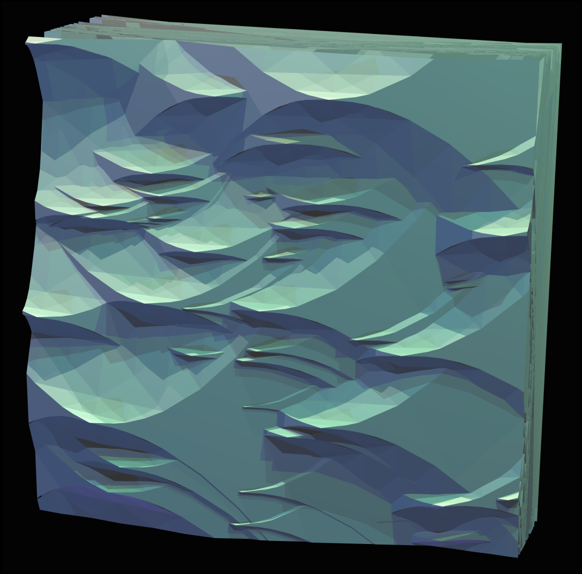
Theorem (Naor-Y.)

If $E \subset H^{2k+1}$, then E can be decomposed into sets E_i so that each ∂E_i has a corona decomposition that approximates ∂E_i by intrinsic Lipschitz graphs.

An intrinsic Lipschitz graph



Bounding the roughness of surfaces

Theorem (Austin-Naor-Tessera, Naor-Y.)

If $k \geq 2$ and $S \subset B \subset H^{2k+1}$ is bounded by an intrinsic Lipschitz graph with bounded Lipschitz constant, then

$$\int_0^1 \frac{\mathrm{vol}(S \bigtriangleup SZ^t)^2}{t^2}\, dt \lesssim 1.$$

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Theorem (Naor-Y.)

If $k \geq 2$ and $S \subset B \subset H^{2k+1}$ is a set such that ∂S has a corona decomposition, then

$$\int_0^1 \frac{\mathrm{vol}(S \bigtriangleup SZ^t)^2}{t^2}\, dt \lesssim \mathrm{area}(\partial S)^2.$$

Open questions

- What happens in H^3? Sets in H^3 can still be decomposed in the same way, but the inequality may not hold.

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- What happens in H^3? Sets in H^3 can still be decomposed in the same way, but the inequality may not hold.
- Uniform rectifiability in $\mathbb{R}^k$ has definitions in terms of singular integrals, β–coefficients, corona decompositions, the big-pieces-of-Lipschitz-graphs property, and many more. We've used corona decompositions to study one class of surfaces in the Heisenberg group – do the rest of the definitions also generalize?