Filling multiples of embedded curves and quantifying nonorientability

Robert Young
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- $n = 3$: If T is a curve in $\mathbb{R}^3$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$. (Federer, 1974)

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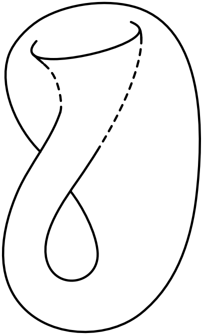
- $n = 3$: If T is a curve in $\mathbb{R}^3$, then $\mathrm{FA}(2T) = 2\,\mathrm{FA}(T)$. (Federer, 1974)
- $n = 4$: There is a curve $T \in \mathbb{R}^4$ such that

$$\mathrm{FA}(2T) \leq 1.52\,\mathrm{FA}(T)$$

  (L. C. Young, 1963)

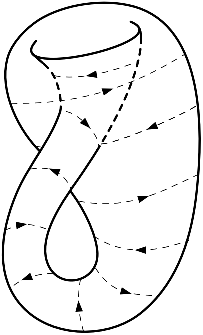
L. C. Young's example

Let K be a Klein bottle



L. C. Young's example

Let K be a Klein bottle and let T be the sum of $2k + 1$ loops in alternating directions.



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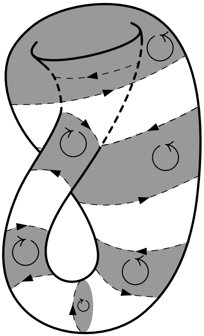

- T can be filled with k bands and one extra disc D
- $\mathrm{FA}(T) \approx \frac{\text{area } K}{2} + \text{area } D$

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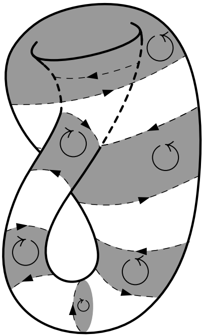

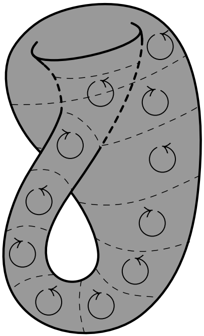

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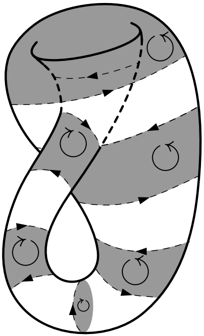

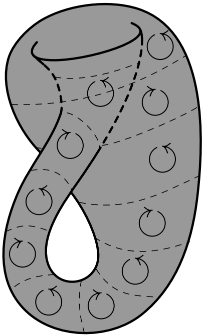

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- $2T$ can be filled with $2k+1$ bands
- $\mathrm{FA}(2T) \approx \text{area } K$— less than $2\,\mathrm{FA}(T)$ by $2\,\text{area } D$!

The main theorem

Q: Is there a $c > 0$ such that $\mathrm{FA}(2T) \geq c\,\mathrm{FA}(T)$?

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Theorem (Y.)

Yes! For any d, n, there is a c such that if T is a d-cycle in $\mathbb{R}^n$, then $\mathrm{FA}(2T) \geq c\,\mathrm{FA}(T)$.

Corollaries

Let $\mathcal{F}_d(\mathbb{R}^N)$ be the set of integral flat chains in $\mathbb{R}^N$. Then:

- If $k > 0$ is a positive integer, the multiply-by-k map $f : \mathcal{F}_d(\mathbb{R}^N) \to \mathcal{F}_d(\mathbb{R}^N)$, $f(T) = kT$ is an embedding.

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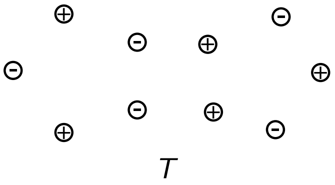
- If T is a mod-k current, then $T \equiv T_{\mathbb{Z}} \pmod{k}$ for some integral current $T_{\mathbb{Z}}$. Consequently, the set of mod-k currents is a quotient of the integral currents.

Proving the theorem in dimension 0

Strategy: If B is a filling of $2T$, then "half of B" fills T.



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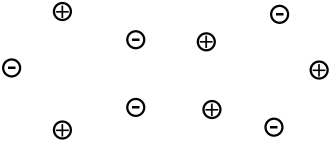
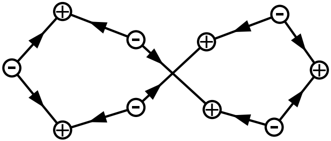


T



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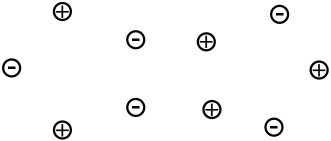

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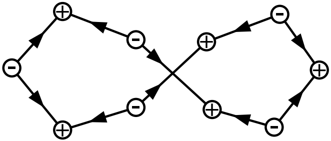

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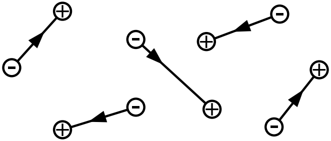
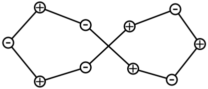


"half of B" is a filling of T

What does “half” mean?

Consider the mod-2 cycle B mod 2.



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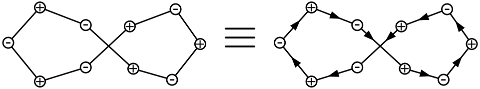

(mod 2)

B mod 2 $\qquad$ P

Then B mod 2 is an orientable closed curve with orientation P.

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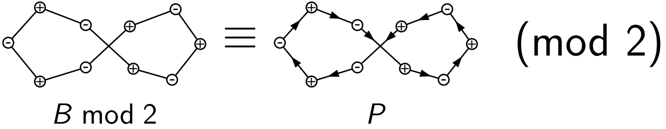

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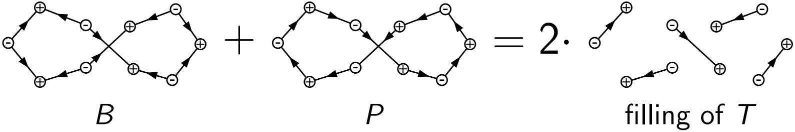
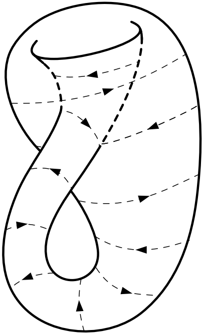


“Half” of the Klein bottle

Let T be a cycle



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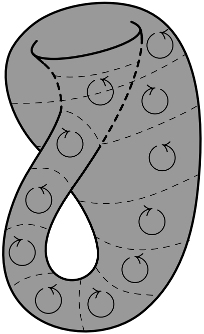

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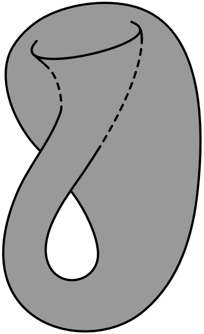
so B mod 2 is a cycle.



B mod 2

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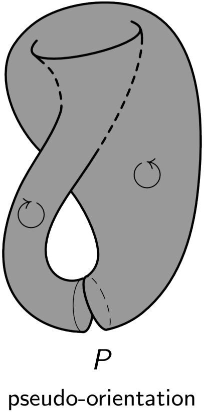

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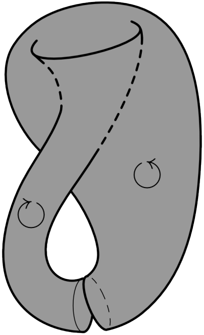
then

$$B + P \equiv 0 \pmod 2$$

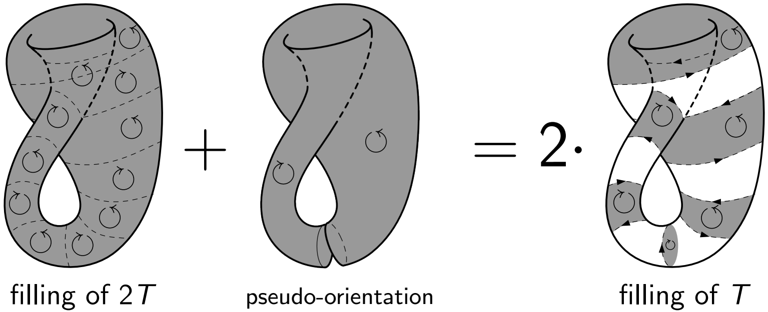
$$\partial \frac{B+P}{2} = \frac{2T+0}{2} = T.$$



P

pseudo-orientation

The Klein bottle, again



Nonorientability

If A is a mod-2 cycle, define the *nonorientability* of A by

$$\mathrm{NO}(A) = \inf\{\operatorname{mass} P \mid P \text{ is an integral cycle and } P \equiv A \pmod 2\}$$

This measures how hard it is to "lift" A to an integral cycle.

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If $\partial B = 2T$, then

$$\mathrm{FV}(T) \le \frac{\mathrm{mass}\, B + \mathrm{NO}(B \bmod 2)}{2}$$

So, to prove that $\mathrm{FV}(T) \lesssim \mathrm{FV}(2T)$, it suffices to show:

Proposition

If A is a mod-2 d-cycle in $\mathbb{R}^n$, then $\mathrm{NO}(A) \lesssim \mathrm{mass}\, A$.

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Strategy:

- Find a mod-2 $(d+1)$-chain such that $A = \partial F$, then lift F to an integral chain $F_{\mathbb{Z}}$. Then $P = \partial F_{\mathbb{Z}}$ is a pseudo-orientation of A.

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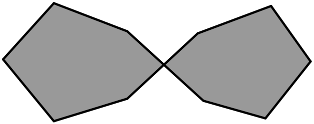
- Generally, F will be non-orientable. We will need to cut F into orientable pieces to get $F_{\mathbb{Z}}$, and $\mathrm{NO}(A)$ measures how much of F we need to cut.

Codimension 1

If A is codimension 1, then A is the boundary of a top-dimensional chain F:



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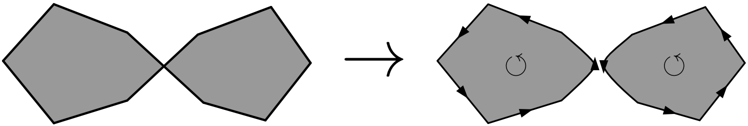
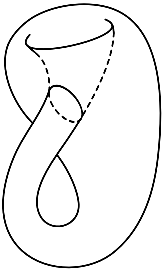


F is orientable, so A is orientable and $\mathrm{NO}(A) = \mathrm{mass}(A)$.

Example: the immersed Klein bottle

A Klein bottle immersed in $\mathbb{R}^3$ has an inside and an outside



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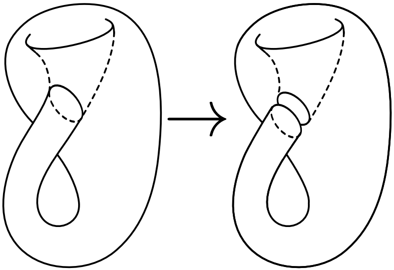

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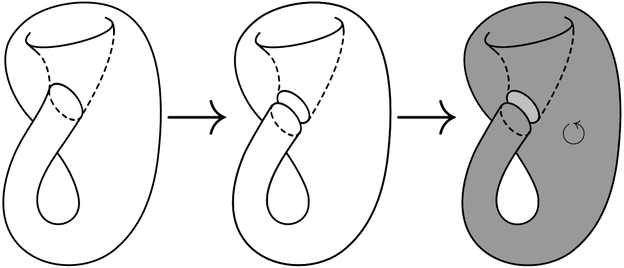

so it is orientable!

Results in low codimension

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Every $(n-1)$–cycle in $\mathbb{R}^n$ is orientable.

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Corollary (Federer)

If T is an integral $(n-2)$–cycle in $\mathbb{R}^n$, then $\mathrm{FV}(2T) = 2\,\mathrm{FV}(T)$.

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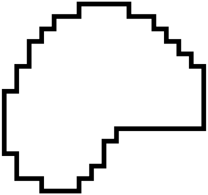
What about higher codimensions?

A simple argument in high codimension

Let A be a mod-2 cellular d-cycle of mass V



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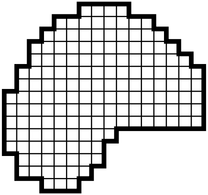

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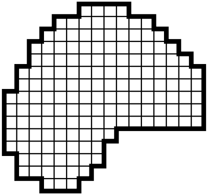

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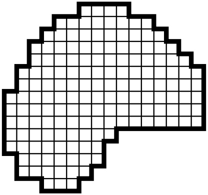

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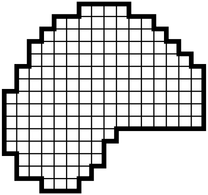

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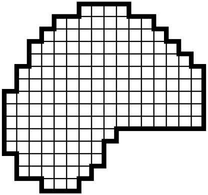

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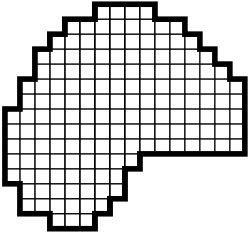
- Orient the cubes at random to get $F_{\mathbb{Z}}$
- $\partial F_{\mathbb{Z}}$ is a pseudo-orientation
- $\mathrm{NO}(A) \lesssim \mathrm{mass}\, \partial F_{\mathbb{Z}} \sim V^{(d+1)/d}$

Bigger cubes



Total boundary: $V^{(d+1)/d}$

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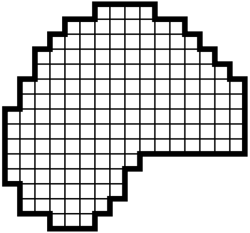

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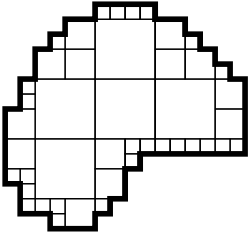
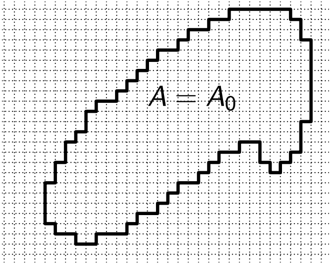


Total boundary: much less

Filling through approximations



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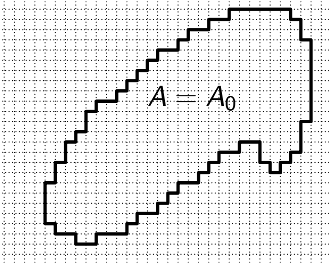

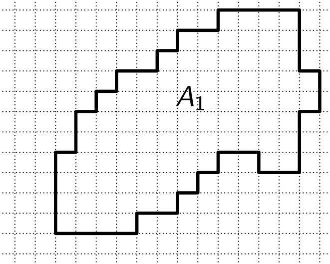

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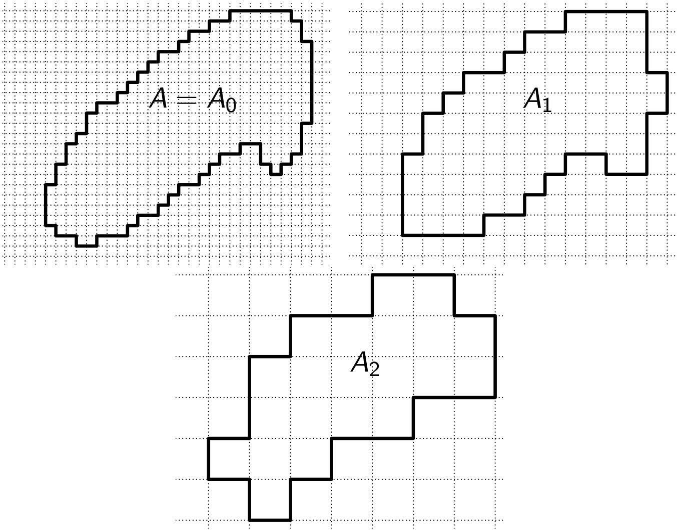

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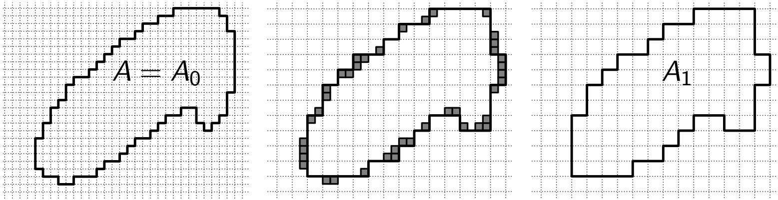

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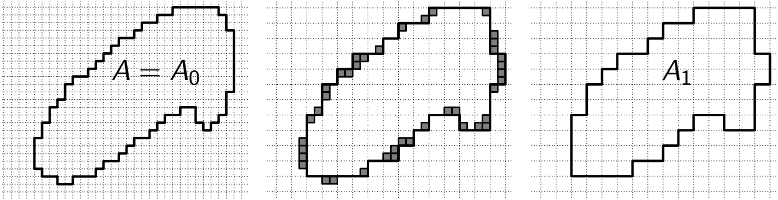

$\sim V$ squares each with perimeter ~ 1

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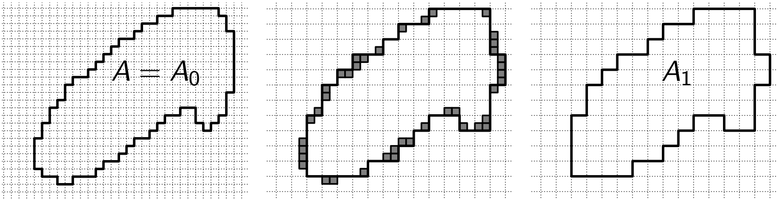

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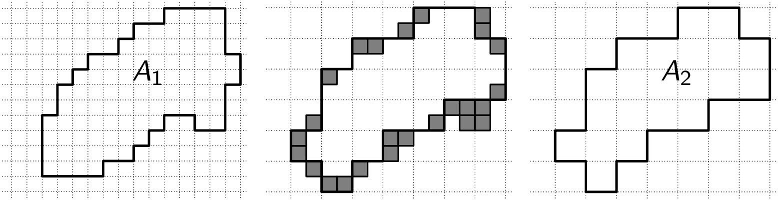
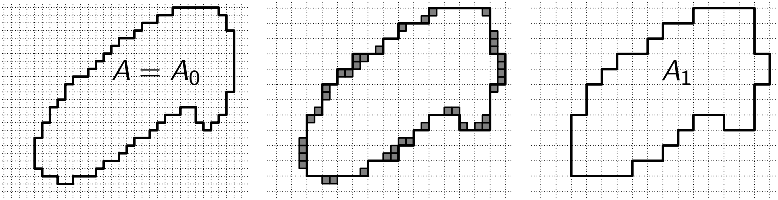
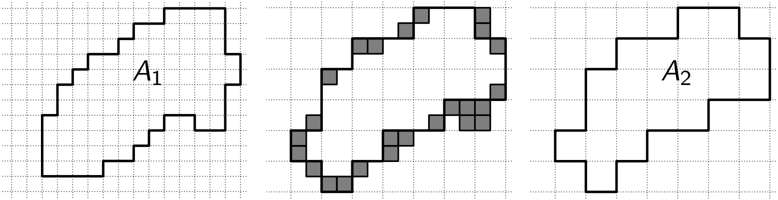


Filling through approximations



$\sim V$ squares each with perimeter ~ 1



$\sim V/2$ squares each with perimeter ~ 2

Filling through approximations

Sketch:

- Approximate A at $\sim \log V$ scales, then connect the approximations.
- We use cubes with total boundary $\sim V$ at each scale.
- Since there are $\sim \log V$ scales, we conclude:

Proposition (Guth-Y.)

If A is a cellular mod-2 cycle with volume V, then it has a pseudo-orientation P such that $\text{mass}\, P \lesssim V \log V$.

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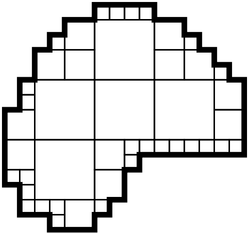
Now we just need to get rid of the log factor!

Getting rid of the log factor



- Choosing orientations randomly is wasteful when A is close to a plane

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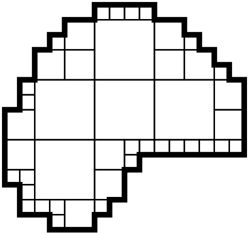

- Choosing orientations randomly is wasteful when A is close to a plane
- But what if A is never close to a plane?

Dealing with fractals

How do we prove the proposition for sets that are close to fractals?

- Show that adding topological complexity adds extra area
- Prove the theorem when A has "low complexity"

Properties of “simple” sets

Definition

A set $E \subset \mathbb{R}^n$ is Ahlfors d-regular if for any $x \in E$ and any $0 < r < \operatorname{diam} E$,

$$\mathcal{H}^d(E \cap B(x,r)) \sim r^d.$$

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A set $E \subset \mathbb{R}^n$ is d-rectifiable if it can be covered by countably many Lipschitz images of $\mathbb{R}^d$.

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But sets that are close to fractals can be regular, rectifiable, but still very complicated!

Uniform rectifiability

Definition (David-Semmes)

A set $E \subset \mathbb{R}^n$ is uniformly d-rectifiable if it is d-regular and there is a c such that for all $x \in E$ and $0 < r < \operatorname{diam} E$, there is a c-Lipschitz map $B_d(0, r) \to \mathbb{R}^n$ which covers a $1/c$-fraction of $B(x, r) \cap E$.

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These sets are close to planes on “most” balls.

Sketch of proof

Proposition

Every mod-2 cellular d-cycle A can be written as a sum

$$A = \sum_i A_i$$

of mod-2 cellular d-cycles with uniformly rectifiable support such that

$$\sum \operatorname{mass} A_i \leq C \operatorname{mass} A.$$

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Proposition

Any mod-2 cellular d-cycle A with uniformly rectifiable support has a pseudo-orientation P with

$$\operatorname{mass} P \leq C \operatorname{mass} A.$$

Open questions

- More generally,

$$\mathsf{FV}(T) \geq c_k \frac{\mathsf{FV}(kT)}{k}.$$

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- More generally,

$$\mathrm{FV}(T) \geq c_k \frac{\mathrm{FV}(kT)}{k}.$$

  Can the c_k be chosen uniformly?
- What does this tell us about the geometry of surfaces embedded in $\mathbb{R}^n$ by a bilipschitz map?