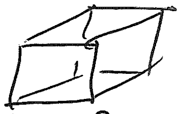
Last time: ~~Topological property: property preserved by homeomorphisms.~~ Q: How do you draw a picture of a topological space?

~~Today: Connectedness.~~



Well, a picture is a function to the plane:
Top. space - what are the continuous functions?

Ex: X with discrete topology - any $f: X \to \mathbb{R}^2$ is continuous.
X with indiscrete top: - only constant fns are cts
(and you saw this in the problem set ~~3~~ - if u, v can't be separated, then cts maps send them to the same pt)
In particular, this shows up in a prop we'll look at today:

Today: Connectedness: ~~Q: Is there a cts~~

Def: ~~X is connected~~ A separation of X is a pair of nonempty open sets A, B s.t. $A \cap B = \emptyset$, $A \cup B = X$. We say that X is connected if it does not have a separation. Equiv: Any subset of X which is open & closed is $\emptyset$ or X.

This is a little abstract at first, but:
~~Prop: The following are equivalent:~~
1) X is not connected $\Leftrightarrow$
2) There is a continuous surjective map $f: X \to \{0,1\} \subset \mathbb{R}$

Pf: ~~Let~~ If A, B separate X, let $f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B. \end{cases}$
This is surjective, $f^{-1}(0) = A$, $f^{-1}(1) = B$ are open, so f is cts.
Converseley, if $f: X \to \{0,1\}$ is cts, surj, then $A = f^{-1}(0)$, $B = f^{-1}(1)$ is a separ

Ex: $\mathbb{R} \setminus \{0\}$ is disconnected bc $A = (-\infty, 0)$, $B = (0, \infty)$ is a sep.
More gen, any subset of $\mathbb{R}$ that is not an interval is disconnected

~~Ex:~~ Prop: Any interval in $\mathbb{R}$ is connected.
Pf: Intermediate Value Theorem.

In fact, let's expand on this a little:
Def: A space X satisfies the Intermediate Value Property if the image of any cts map $f: X \to \mathbb{R}$ is an interval.
($S \subset \mathbb{R}$ is an interval if $[a,b] \subset S$ $\forall a, b \in S$ with $a \leq b$).

Prop: X is connected $\iff$ X satisfies the IVP.

Pf: If X is disconnected, $\exists f: X \to \{0,1\}$ cts & surj, so ~~a non-cts~~ f does not have IVP.

($\Leftarrow$) ~~Suppose X is connected~~, ~~$f: X \to \mathbb{R}$ is cts. Suppose $f(X)$~~ ~~is not an~~ OTOH, suppose X doesn't satisfy IVP. Then $\exists f: X \to \mathbb{R}$ cts s.t. $f(X)$ is not an interval. That is, $\exists$ a, ~~$b \in f(X)$ s.t. $a < c < b$~~, $c \in \mathbb{R}$, $a < c < b$ s.t. $a, b \in f(X)$ but $c \notin f(X)$.

Consider $A = f^{-1}((-\infty, c))$, $B = f^{-1}((c, \infty))$. Then $a \in A$, $b \in B$, so A, B nonempty, $A \cap B = \phi$, and $A \cup B = X$. So A, B separate X //

So: $\mathbb{R}$ is connected. ~~The~~ ~~Special cases of following:~~ Once we know one connected set, we can build more: ~~Two lemmas~~.

Prop: If X is connected, $f: X \to Y$ cts, then $f(X)$ is connected.

Pf: ~~Let~~ If A, B separate $f(X)$, then $f^{-1}(A), f^{-1}(B)$ separate ~~$f(X)$~~ X. //

(Alternatively, if $g: f(X) \to \mathbb{R}$ is cts, then $g(f(X)) = (g \circ f)(X)$. Since $g \circ f$ is cts, X connected, $g(f(X))$ is an interval.) So ~~in summary: TFAE: - X is connected - Any subset of X which is open & closed is ϕ or X - X satisfies IVP~~

~~How do we prove that a set is connected? One way~~ tools: is based on the following

~~Lemma~~: If $Y = C \cup D$ where C, D are disjoint open sets and if $X \subset Y$ is connected, then $X \subset C$ or $X \subset D$.

Pf: ~~If~~ Consider $A = X \cap C$, $B = X \cap D$. These are disjoint open subsets of X and $A \cup B = X$. Since X is connected, either $A = \phi$ or $B = \phi$ $\Rightarrow X \subset C$ or $X \subset D$ //

Thm: Suppose $Y = \bigcup_{\alpha \in A} X_\alpha$, where ~~$X_\alpha$ is connect~~ $p \in Y$ and $\forall \alpha \in A$, X_α is connected and $p \in X_\alpha$. Then Y is connected.

Pf: Suppose Y ~~[illegible]~~ $= C \cup D$ where C, D open, disjoint. Suppose $p \in C$. ~~Then each~~ $\forall \alpha \in A$, lemma implies $X_\alpha \subset C$ or $X_\alpha \subset D$. Since $p \in C$, $X_\alpha \subset C$ $\forall \alpha \in A$ $\Rightarrow$ $Y \subset C$. So $C = Y$, $D = \phi$ //.

~~How do... And we~~ ~~can~~ Together, this thm, this ~~prop~~ gives a lot of examples.

Def: X is path-connected if $\forall x, y \in X$, $\exists f: [0,1] \to X$ cts such that $f(0) = x$, $f(1) = y$.

Prop: If X is path-connected, it is connected.

Pf: ~~$\forall y \in X$~~ Let $x \in X$. $\forall y \in X$, let $f_y: [0,1] \to X$ s.t. $f_y(0) = x$, $f_y(1) = y$. ~~Then~~ Let $\gamma_y = f_y[0,1]$.

Then γ_y is connected $\forall y \in X$, and $\bigcup_{y\in X} \gamma_y = X$. //

→ This lets us def connected cpts.

But! connected $\not\Rightarrow$ path-connected.

Ex: Topologist's sine curve.

$$S = \{(x, \sin(\tfrac{1}{x})) \mid x > 0\}$$
$$\bar{S} = S \cup \{(0,y) \mid y \in [-1,1]\}$$

Ex: ~~This is not path-connected~~

~~But clear~~ S is connected (cts image of $(0,\infty)$)

~~and $\bar S \setminus S$ is connected~~

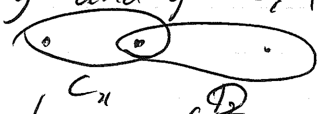
Suppose $\bar S = C \cup D$, C, D disjoint, open.

Then $S \subset C$. But C is closed, so $\bar S \subset C \Rightarrow D = \emptyset$. //

Def: <u>Connected components</u>:

Let X be a space, $x \in X$. ~~The connect~~ We can define an equivalence relation $x \sim y$ if $\exists$ a connected subset C s.t. $x, y \in C$. Then:

- $\forall x$, $x \sim x$ - if $x \sim y$, then $y \sim x$.
- if $x \sim y$ and $y \sim z$, then $x \sim z$.



$C \cup D$ is connected, $x, z \in C \cup D$.

The equivalence classes of $\sim$ are called <u>connected components</u>

~~Prop~~ (i.e. $\forall x \in X$, the connected component of x is the set $C_x = \{y \mid y \sim x\}$) - C_x is connected.

~~Ex: C_x is connected.~~

But $\bar S$ is not path connected.

Pf: Let $p = (0,0)$, $q = (1, \sin 1)$. Suppose $f : [0,1] \to \bar S$ is a cts map s.t. $f(0) = p$, $f(1) = q$.

~~{This spends~~ so Let $L = \{(0,y) \mid y \in [-1,1]\}$. How do we find a contradiction?

{ This curve ~~prob~~ spends some time in L, some time in S.

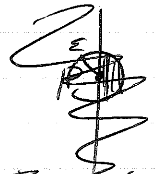
The time in L is fine - def cts maps to L. Likewise S.

The problem will happen when we <u>switch</u> from L to S. So we track that down.

Let $K = f^{-1}(L) \subset [0,1]$ ~~L is closed, so K is closed~~.

Since K is bounded, it has a least upper bound.

Let $b = \sup K$. Since C is closed, K is closed, so $b \in K$.
On ~~the other But if~~ So $b \neq 1$. ~~Further, if $b < c < 1$,~~
~~then $f(c) \in S$.~~ Further, if $b < c < 1$, then
~~Let $b < c \in (b,$~~ ~~Let~~ $f(c) \in S$.



~~Let $\varepsilon > 0$ Let $c \in (b, 1)$.~~
~~Let $p = f(b)$~~
Let ~~$\varepsilon > 0$~~ s.t.
~~$B(p, \varepsilon) \cap S$ is disconnected~~
~~If~~ Then $\exists$ ~~$\delta > 0$~~ s.t. ~~$f([b,c]) \subset B(p,\varepsilon)$~~
~~Let~~
if ~~$|b - c| < \delta$~~, then
$f(c) \in$ ~~$B(p, \varepsilon)$~~.

~~But~~
~~Reparametrize so $b = 0$, $f(t)$~~

Reparametrize so $b = 0$. Let $(x(t), y(t)) = f(t)$. — then
x, y cts, $x(0) = 0$, $x(\frac{1}{n}) > 0$ $\forall n$. Consider $f(\frac{1}{n})$ — $f(\frac{1}{n}) \to f(0)$
by cty.

~~By IVT, $\forall n$, $\exists$ $t_n \in (0, \frac{1}{n})$ s.t. $x(t_n) =$~~
For each n, $\exists$ a_n s.t. $0 < a_n < x(\frac{1}{n})$
and $\sin(\frac{1}{a_n}) = (-1)^n$.
By IVT, $\exists$ $0 < t_n < \frac{1}{n}$ s.t. $x(t_n) = a_n \Rightarrow y(t_n) = (-1)^n$.

Then $t_n \to 0$, but $y(t_n)$ diverges! $\times$ //.

Overflow: Questions Compactness. Product topology?

One important result:
Thm: $\mathbb{R} \not\cong \mathbb{R}^2$.
Pf: ~~Let $p \in \mathbb{R}$. Then $\mathbb{R} \setminus \{p\}$ is disconnected.~~
~~Let $q \in \mathbb{R}^2$~~
If $f : \mathbb{R} \cong \mathbb{R}^2$ then $\forall p \in \mathbb{R}$,
$\mathbb{R} \setminus \{p\} \cong \mathbb{R}^2 \setminus \{f(p)\}$
But $\mathbb{R} \setminus \{p\}$ is disconnected and
$\mathbb{R}^2 \setminus \{p\}$ is connected. //.