Slides: Cayley graphs, scaling, distribution (2018-10-0

Okay, where were we? We defined a group:

$$\mathbb{H} = \left\{ \begin{pmatrix}1 & x & z\\ & 1 & y \\ & & 1\end{pmatrix} \;\middle|\; x,y,z\in\mathbb{R}\right\}.$$

We'll write $\mathbb{H}=\mathbb{R}^3$ with operation $(x,y,z)(x',y',z') = (x+x', y+y', z+z'+xy')$, ~~t~~ This is a Lie group, and $\forall t>0$, $s_t(x,y,z) = (tx, ty, t^2z)$ is an isomorphism.

Goal: Find a left-invariant metric on $\mathbb{H}$ that is compatible w/ scalings. This will be a sub-Riemannian metric — a metric based on a distribution of tangent planes and a quadratic form on those planes (like Riemannian, except there's only some subset of vectors w/ finite length and rest have infinite length).

First, some left-invariant fields. Let $X=(1,0,0)$, $Y=(0,1,x)$, $Z=(0,0,1)$ (~~mult by (1,0,0)~~ three gens correspond to elementary matrices, mult by each takes you in one of these dirs.)

Let $H = \langle X, Y\rangle \subset T\mathbb{R}^3$ be the horizontal distribution.

Let $g = dx^2+dy^2$. ~~The restriction of~~ Then g is left-invariant ~~(and why not)~~

Like Riem metric, distance is the length of shortest curve. Since only some ~~curves are fin~~ dirs are finite, shortest curve means shortest curve ~~t~~ tangent to horiz dist:

Def: A curve $\gamma:[0,1]\to\mathbb{H}$ is horizontal if $\gamma'(t)\in H$ for almost every $t\in[0,1]$. If $\gamma(t)=(x(t),y(t),z(t))$ define $\ell(\gamma)=\int_0^1\sqrt{x'(t)^2+y'(t)^2}\,dt$.

$\forall v,w\in\mathbb{H}$, let $d(v,w) = \inf_{\substack{\gamma:[0,1]\to\mathbb{H}\\ \gamma(0)=v,\ \gamma(1)=w}} \ell(w)$.

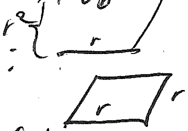
Okay, what does this mean? Well, one way to understand is this:

If $\gamma=(x,y,z)$ is horizontal, then $(x',y',z') = x'\cdot X + y'\cdot Y$ $\iff$ $z'=xy'$. So the z coord is determined by ~~the~~ the projection to the xy-plane. ~~Def~~ Conversely, given a curve in the plane, ~~Prop: Given~~ it lifts to a horizontal curve: Let $\pi:\mathbb{H}\to\mathbb{R}^2$, $\pi(x,y,z)=(x,y)$.

Prop: Let $\lambda=(x(t),y(t))$ be a curve in $\mathbb{R}^2$, let $z_0\in\mathbb{R}$. There's a unique ~~horiz~~ horizontal curve $\tilde\lambda:[0,1]\to\mathbb{H}$ s.t. ~~$\tilde\lambda(t)=(x(t),y(t),z_0$~~ $\pi\circ\tilde\lambda=\lambda$ and $\tilde\lambda(0)=(x(0),y(0),z_0)$.

This is given by $\tilde{\lambda}(t) = \left(x(t), y(t), z_0 + \int_0^t x(t)\, y'(t)\, dt\right)$.
Call this the <u>lift</u> of λ. Note that $\ell(\tilde{\lambda}) = \ell(\lambda)$.

~~Ex~~: If λ is a closed curve in $\mathbb{R}^2$, then $\tilde{\lambda}(1) - \tilde{\lambda}(0) = \left(0, 0, \int_0^1 x(t)\, y'(t)\, dt\right) = (0, 0, \text{signed area of } \lambda)$.

So:  — we can travel up/down in $\mathbb{H}$ by going around a closed curve in $\mathbb{R}^2$.

~~Properties of d~~: So: That's enough to give ~~us a couple of estimates~~:

- $d(v,w) < \infty$ $\forall v, w \in \mathbb{H}$. Once we know that, we can say:
- d is a metric
- $\forall g \in \mathbb{H}$, $d(gv, gw) = d(v,w)$
- $\forall t > 0$, $d(s_t(v), s_t(w)) = t\, d(v,w)$
- $d(0, (x,y,z)) \approx \max\{|x|, |y|, \sqrt{|z|}\}$ — Hausdorff dimensions — $\dim(\mathbb{H}) = 4$, $\dim = 2$, $\dim V = 3$

Today: Surfaces in $\mathbb{H}$. (Standing exercise)
First: smooth.

~~Not hard to show~~ Prop: There are no smooth horizontal surfaces in $\mathbb{H}$. (Not hard)

Pf: Suppose Σ is a horizontal surface. Let $p \in \Sigma$. Then $T\Sigma_p = H_p$ ~~But then~~ So a small disc around p projects to a disc in $\mathbb{R}^2$. But that's a horizontal closed curve, so its projection must have area 0 ✗

So, any smooth horizontal map $D^2 \to \mathbb{H}$ has derivative of rank ≤ 1 everywhere.

Let's weaken the condition a little: Lipschitz. (def)

Are there Lipschitz surfaces in $\mathbb{H}$? No.

Thm (Pansu): If $U \subset \mathbb{R}^n$ is open, $f: U \to \mathbb{H}$ is Lipschitz, then f is Pansu-differentiable: for a.e. $x \in U$, $\exists$ a homomorphism $h: \mathbb{R}^2 \to \mathbb{H}$ with horizontal image s.t.

$$\lim_{\|v\| \to 0} \frac{d(f(x+v), f(x) \cdot h(v))}{\|v\|} = 0$$

~~(i.e., locally it~~ (i.e., if you zoom in on the image, it looks ~~like a homo~~ not just like a linear map, but like a homomorphism.)

In particular: - Lipschitz curves are horizontal, ~~and there's no Lipschitz embedding $f: D^2 \to \mathbb{H}$~~.

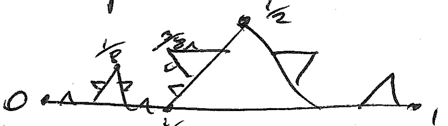
- Any Lipschitz map $f: D^2 \to \mathbb{H}$ has a derivative with rank ≤ 1 a.e., so area$(\pi \circ f) = 0$. For example, factor through tree, disc, etc.
- In fact, (and this takes a lot of work to show)

Thm (Wenger–Y): ~~Any Lipschitz map $f: D^2 \to \mathbb{H}$ factors~~
If $f: D^2 \to \mathbb{H}$ is a Lipschitz map, then there is a metric tree T and maps $g: D^2 \to T$, $h: T \to \mathbb{H}$ s.t. $f = h \circ g$.

Let's try to weaken it even further.

Def: ~~Let~~ $0 < \alpha \leq 1$. A map $f: X \to Y$ is α-Hölder if $\exists\, C > 0$ s.t.
$$d(f(x_1), f(x_2)) \leq C\, d(x_1, x_2)^\alpha$$

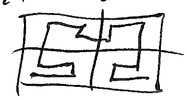
This comes up in a lot of fractals: Koch snowflake:



This is not Lipschitz – $d(\gamma(t), \gamma(t + \frac{1}{4^n})) \approx \frac{1}{3^n} \approx$ ~~...~~ $|t - (t + \frac{1}{4^n})|^{\frac{\log 3}{\log 4}}$

But it's $\frac{\log 3}{\log 4}$-Hölder: ~~$\left(\frac{1}{3^n}\right)^{\frac{\log 3}{\log 4}} = \ldots$~~
$$\left(\frac{1}{4^n}\right)^{\frac{\log 3}{\log 4}} = \left(\frac{1}{3}\right)^n$$

Exercise: Smaller ~~H~~ exponent $\Rightarrow$ wilder maps: ~~Brownian motion~~ ~~space~~



Hilbert curve is $\frac{1}{2}$-Hölder,
Brownian motion is $\frac{1}{2}$-Hölder,
space-filling: $\frac{1}{3}$-Hölder.

Prop: ~~If X has Hausdorff~~
If f is α-Hölder, then ~~Haus~~-$\dim_{Haus}(f(X)) \leq \frac{\dim_{Haus}(X)}{\alpha}$

Thm (Pansu, Gromov): Any embedded surface in $\mathbb{H}$ has Hausdorff dimension ≥ 3. (In fact, any set that ~~separates~~ disconnects $\mathbb{H}$)

Cor: ~~(Gromov)~~ There is no α-Hölder embedding of D^2 in $\mathbb{H}$ for $\alpha > \frac{2}{3}$. (Züst): Any α-Hölder map with $\alpha > \frac{2}{3}$ factors through a tree)

~~Hm~~ How much do we need to relax to get a surface?
Well, the vertical plane is parametrized by a $\frac{1}{2}$-Hölder map – z-axis is dim 2, so a map to the plane has exponent $\leq \frac{1}{2}$.

And here's ~~some~~ an idea of why it might be hard:
~~Höld~~ When $\alpha > \frac{1}{2}$, α-Hölder maps to $\mathbb{R}^2$ lift.
(Remember the signed area pic)

Signed area is well-defined for α-Hölder maps when $\alpha > \frac{1}{2}$:

~~So, if $\alpha > \frac{1}{2}$, an α-Hölder map from~~ $f: D^2 \to H$ has ~~the property that $\forall$ closed curve~~ $\gamma \subset D^2$, ~~$\pi \circ f(\gamma)$ has~~ γ is horizontal

Prop: If $\alpha > \frac{1}{2}$, $\gamma: [0,1] \to \mathbb{R}^2$ is α-Hölder, $z_0 \in \mathbb{R}$, then $\exists!$ lift $\tilde{\gamma}$ s.t. $\pi \circ \tilde{\gamma} = \gamma$, $\tilde{\gamma}(0) = (x(0), y(0), z_0)$.
If $\gamma(0) = \gamma(1)$, then $\tilde{\gamma}(1) - \tilde{\gamma}(0) =$ signed area of γ.

In particular, if $\tilde{\gamma}: S^1 \to H$ is α-Hölder, then $\pi \circ \tilde{\gamma}$ has signed area 0.

So, if $\alpha > \frac{1}{2}$, then constructing a α-Hölder map $D^2 \to H \iff$ constructing a map $D^2 \to \mathbb{R}^2$ s.t. the image of any closed curve has signed area 0.

Seems hard, right?

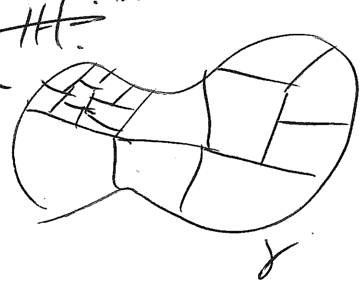
Thm (Wenger-Y): If $\alpha < \frac{2}{3}$, then any continuous map $f: D^2 \to H$ can be approximated uniformly by α-Hölder maps!

Let me try to sketch one such map:

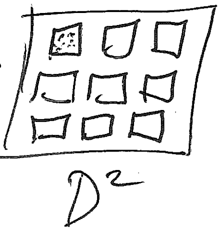
Facts: - $\delta_{H^3}(n) \sim n^3$
- Consequently $\exists\, C > 0$ s.t. $\forall$ suff large k, any closed curve of length L can be subdivided into ck^3 closed curves of length L/k. (Approximate by a curve in Cayley graph, reduce, get subdivision.)

Let k be large.

Map: 1. Pick a closed horizontal curve $\gamma: S^1 \to H$ of length 1. Define $f(x) = \gamma(x)$ for $x \in \partial D^2$.



2. Subdivide γ into ck^3 curves of length $\frac{1}{k}$.
3. ~~Draw Choose~~ Draw ck^3 disjoint squares in D^2 with side $\frac{1}{\sqrt{2ck^3}}$
4. Send the boundary of each square to one of the small curves.



D^2

5. Extend to a Lipschitz map on the region between.
6. Repeat for each smaller curve and square.

Now: At the nth generation, squares of side $\left(\frac{1}{\sqrt{2ck^3}}\right)^n$ go to curves of length $\left(\frac{1}{k}\right)^n$. Exponent

$$\frac{\log\left(\frac{1}{k}\right)^n}{\log\left(\frac{1}{\sqrt{2ck^3}}\right)^n} = \frac{-n \log k}{-n \log \sqrt{2ck^3}} \to \frac{2}{3}$$

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