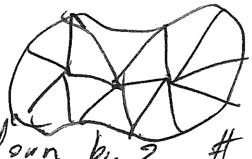
Pf: Remember the idea – to pass from cone to space, we need to express properties in terms of finitely many points.

So: let's say that $X$ is $k$-triangulable if $\forall$ suff. large $L>0$, $\forall$ ~~curves $\gamma$: $S$~~ closed curves $\gamma : S^1 \to X$ of length $\le L$, we can ~~decompose $\gamma$ into~~ fill $\gamma$ with a triangulation with $\le k$ triangles, each of length $\le L/2$.



~~If $X$ is~~ If $X$ is $k$-triangulable for some $k$, then we can repeat the process. Everytime, length goes down by 2, # of triangles up by $k$, so this eventually results in a disc with polynomial area.

Further, $k$-triangulable just relies on finitely many pts!

So let's do this! Suppose $X$ is not $k$-triangullable $\forall k$. Then $\exists$ curves $\gamma_k : S^1 \to X$ with $\ell(\gamma_k) \to \infty$ s.t. $\gamma_k$ has no $k$ triangulation. Consider $X_\omega = \lim_\omega (X, \frac{d}{\ell(\gamma_k)}, \gamma_k(0))$. This is a cone of $X$,

42. $\gamma_\omega = \lim_\omega \gamma_k : S^1 \to X_\omega$ is unit speed, length 1, and by assumption, $\exists$ $\beta : D^2 \to X_\omega$ filling $\gamma$. $\beta$ is unif cts, so a suff fine triangulation of $D^2$ gives a triang of $\gamma_\omega$ into triangles with length $\le 1/3$. The vertices of this triang form a triangulation ~~is some $\gamma_k$~~ ~~$X$~~ all suff. large $\gamma_k$ X.

2018-11-15

Last time: ~~Def~~ Wanted to compare simple connectivity, filling invariants.

~~Thm (Gromov)~~ Def: $k$-triangulability: Any closed curve $\gamma$ of length $L \ge L_0$ can be divided into $k$ triangles of length $\le L/2$. Two consequences:
- If $X$ is $k$-triangulable, then $FA_X \lesssim n^{\log_2 k}$
- If $X$ is $k$-triangulable for $L_0 = 0$, then $\pi_1(X) = 0$.

~~Let $G$ be a f.p. group.~~

Thm (Gromov): If every asymp cone of ~~$G$~~ $X$ is simply connected and $FA_X$ ~~$FA_X$~~ $(n) < \infty$ $\forall n$, then ~~$FA_G(n)$~~ $FA_X \lesssim n^k$ for some $k$.

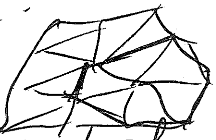
Pf: Suppose ~~$G$~~ $X$ is not $k$-triangulable for any $k$. Then $\exists$ constant speed curves $\gamma_1, \gamma_2, \ldots$ ~~$g$~~ $: S^1 \to$ ~~$X$~~ $X$ ~~with s.t. $\gamma_i$ can~~ with length $\to \infty$ so that $\gamma_i$ can't be subdivided into ~~even two~~ $i$ triangles of length $\frac{\ell(\gamma_i)}{2}$.

Let $C = \lim_{\omega}(X, \frac{d}{\ell(\gamma_i)}, \gamma_i(0))$. Then $[\gamma_i]: S^1 \to C$ is a closed unit-speed curve. Let $\beta: D^2 \to C$ be a null-homotopy.
Then $\beta$ is uniformly continuous – so there is a triangulation $\tau$ of $D^2$ such that ~~the image of every edge has~~ ~~the diameter of~~ every edge has diameter $< 1/12$.
~~Let~~ Label $\beta(\tau^{(0)}) = \{v^1, \dots, v^j\}$, $v^1 = [v_i^1], \dots, v^j = [v_i^j]$

By passing to a subsequence, so that $d(v^j, v^k) = \lim_{i \to \omega} \frac{d(v_i^j, v_i^k)}{\ell(\gamma_i)} \le \frac{1}{12}$. When $i$ is suff large, then $v_i^j$s are the vertices of a triangulation of $\gamma_i$ with every edge of length $\le \frac{\ell(\gamma_i)}{6}$

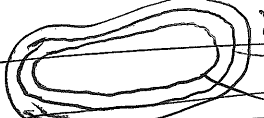
Conversely: ~~this is harder~~ Thm (Papasoglu) (conj. by G.): If $X$ is a simply-connected simplicial complex, and $FA_X(n) \le Cn^2$, then $X$ is $k$-triangulable for some $k$ depending on $C$.
Therefore, $\pi_1(\mathrm{Cone}_\omega X) = 0 \ \forall \omega$.

Pf is more complicated but to give a brief sketch:
Idea: Consider a minimal disc $\beta$ filling a curve $\gamma$.
We can discretize so that $\gamma$ is an edge path, $\beta$ is a triangulation of the disc:



The assumption ~~that~~ on $FA_X$ gives us some bounds on the disc - in particular, since $\beta$ is minimal, any sub-disc is minimal.

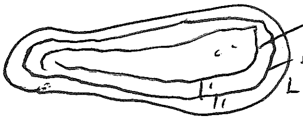
~~How to subdivide Shelling~~ 1. Cut into shells:


~~This is too many shells for our subdiv,~~ but if $l_i = \ell(\gamma_i)$, then $\sum l_i \approx$ Area

Idea: Cut into shells, subdivide the shells:


But we've seen that this fails ~~in the~~ when $\delta_X$ is large ~~- what goes wrong~~?
~~Two things: 1 - This depends on the length of the shells~~
~~2 - This depends on the number of shells.~~
Where do we use the hypothesis? Two places
1 - ~~We need a bound on the # of shells~~
We need a bound on the length of the shells
(Note - ~~the~~ each shell is part of the subdiv, so if the shells are too long, problems).




Then $\sum l_i \le 3\,\mathrm{Area}(B) \le 3CL^2$ (each vert is in at most one shell)

Total length is quadratic, but we only want shells spaced $L/20$ apart.

$\exists\, a \in \{1, \dots, L/20\}$ s.t.

$$l_a + l_{a+\frac{L}{20}} + l_{a+\frac{2L}{20}} + \dots \le \frac{3CL^2}{L/20} = 60CL.$$

2 - This process <u>stops</u>: ~~we don't want too many shells.~~

If $l_{a+\frac{iL}{20}} < L/2$, then we can stop. This happens for some $i \le 120C$.

So we can cut $B$ into bddly many shells with linear perimeter, then cut each shell into bddly many triangles. ✓

Note: This is very sensitive to hypotheses:

Thm (Sapir-Olshanskii): $\exists$ a group $G$ with $\delta_G(n) \lesssim n^2 \log n$ s.t. no asymptotic cone of $G$ is simply connected.

So $n^2$ is special here.

Open questions: ~~Suppose $(D^2, g_i)$ is a sequence of metrics s.t. $FA_{(D^2,g_i)}(L) \le CL^2$, $l(\partial D^2)=1$. By these arguments, $D_\omega = \lim_\omega (D^2, g_i)$ is simply connected, and $D_\omega = f(D^2)$ for some map $f$ st~~

Def: A space $X$ is Lipschitz 1-connected if $\exists\, C>0$ s.t. $\forall L\ \forall \gamma: S^1 \to X$ $L$-Lipschitz, $\exists$ a $CL$-Lip extension $\beta: D^2 \to X$. This immediately implies $FA_X(L) \lesssim L^2$.

Q: ~~Suppose $\delta_G(n) \le Cn^2$, Is $\mathrm{Cone}_\omega G$ Lipschitz. Is $G$ Lipschitz. $G$ acts geometrically on a simpl~~

Suppose $X$ is simply connected, $FA_X(L) \lesssim L^2$. Is $X$ Lipschitz 1-connected? Is $\mathrm{Cone}_\omega X$ Lipschitz 1-connected?

~~Actually~~ ~~Actually its~~ This sounds ~~like a GGT~~ hard question, but it's really a ~~reduce to a~~ question about discs: Suppose $(D^2, g_i)$ is a seq of metrics s.t. $FA_{(D^2,g_i)}(L) \le CL^2$, $l(\partial D^2)=1$. ~~By Papasoglu,~~ By Papasoglu, $D_\omega = \lim_\omega (D^2, g_i)$ is simply connected, and there's a surjective map $f: D^2 \to D_\omega$ that restricts to a degree-1 map on $\partial D^2$. Can we make $f$ Lipschitz?

What can we say about the geometry of $D_\omega$? (break)

Loose ends: Heisenberg group:

How does this picture work in Heis? Recall:

$\mathbb{H}^3 = \left\{\begin{pmatrix}1 & x & z\\ & 1 & y\\ & & 1\end{pmatrix}\right\}$. $H^3 = \langle X, Y, Z \mid [X,Y] = Z$, all others commute $\rangle$

$\mathbb{H}^3_{\mathbb{Z}} = \langle X, Y, Z \mid$ " " $\rangle$ by BCH (but that doesn't work for every nilp gp.) $\delta_{\mathbb{H}^3}(n) \sim n^3$

$\mathbb{H}^5 = \left\{\begin{pmatrix}1 & x & u & z\\ & 1 & & y \\ & & 1 & v\\ & & & 1\end{pmatrix}\right\}$ $H^5 = \langle X, Y, U, V, Z \mid [X,Y]=[U,V]=Z,$ all others commute$\rangle$

$\mathbb{H}^5_{\mathbb{Z}} =$ " "

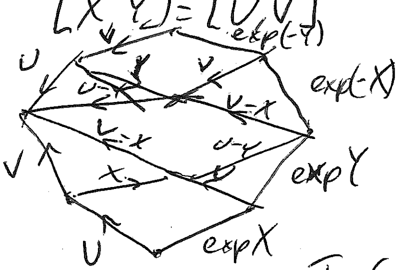
But $\delta_{\mathbb{H}^5}(n) \sim n^2$. Why? $\mathrm{Cone}_\omega \mathbb{H}^3 = (\mathbb{H}^3, d_c)$ has only surfaces of Haus. dim 3. $\mathrm{Cone}_\omega \mathbb{H}^5$ has many surfaces of Haus-dim

$H^5 = \langle X, Y, U, V\rangle \oplus \langle Z \rangle$
horizontal vertical
$= W_1 \oplus W_2$

$\mathrm{Cone}_\omega(\mathbb{H}^5) = (\mathbb{H}^5, d_c)$ with horizontal bundle $W_1$

Recall: one reason for no surfaces in $\mathbb{H}^3$ was Pansu differentiability:
Every ~~horizontal~~ Lipschitz map $f: \mathbb{R}^2 \to \mathbb{H}^5$ is diff. a.e. and $Df_p : \mathbb{R}^2 \to H^5$ is a Lie algebra homomorphism with $Df_p(\mathbb{R}^2) \subset W_1$. There weren't many of these in $\mathbb{H}^3$ bc. horiz bundle generated by two noncommuting vecs, but now:
$[X, U] = 0 = [aX + bU, bY - aV] =$ etc.

And: If $A, B \in W_1$, $[A,B]=0$, then $\exp\langle A, B\rangle$ is an abelian subgroup of $\mathbb{H}^5$ with tangent plane $\langle A, B\rangle$ everywhere — a horizontal surface. There are enough of these to fit together:
$[X, Y] = [U, V]$




So this closed curve bounds a horizontal disc of Haus. dim 2.

Thm (Gromov, Allcock): Every horizontal curve in $\mathbb{H}^5$ bounds a horizontal disc.
In fact, $(\mathbb{H}^5, d_c)$ Lipschitz 1-connected $\Rightarrow \delta_{\mathbb{H}^5} \precsim n^2$

---

General case? If $G$ is nilpotent, then $\mathrm{Cone}_\omega G \cong \mathbb{R}^k$ ~~for some k~~ (with a sub-R metric) for some $k \Rightarrow \delta_G \precsim n^\alpha$ for some $\alpha$.
What $\alpha$? If $G$ not finite or virtually $\mathbb{Z}$ then $\delta_G \succsim n^2$.
Thm (Gersten-Holt-Riley): If $G$ is nilpotent of step $k$ (i.e. $G^{(k+1)} = 0$) then $\delta_G \precsim n^{k+1}$.
Thm (Baumslag-Bridson-Miller-Short): If $F_{n,k} = F_n / F_n^{(k+1)}$, then $\delta_{F_{n,k}} \succsim n^{k+1}$.

Ideas are similar to what we've seen: Roughly if $G$ is step $k$, then $\mathrm{Cone}_\omega G$ has surfaces of Hausdorff dimension $k+1$.
If $G = F_{n,k}$, then $G$ has central extensions that count ~~measure~~ the number of relations used.

The thing about these methods: these both always give polynomial bounds. Are DF's of nilp gps always asymp to a poly? No!

Thm (Wenger): Let $G = F_{6,2}(a_1,\dots,a_6) \times F_{6,2}(b_1,\dots,b_6)$

$[a_i,a_j]=[b_i,b_j]\ \forall i,j$, $[a_1,a_2][a_3,a_4][a_5,a_6]=1$.
Then $n^2 \lneq \delta_G \lesssim n^2 \log n$.

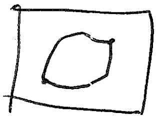
Why? Lie algebra $\mathfrak{g} = \underbrace{\langle A_1,\dots,A_6,B_1,\dots,B_6\rangle}_{W_1} \oplus \underbrace{\langle [A_i,A_j]\rangle / \alpha}_{W_2}$

where $\alpha = [A_1,A_2]+[A_3,A_4]+[A_5,A_6]$

This is a quotient $\mathfrak{g} = \mathfrak{h}/\alpha$ where $\mathfrak{h} = W_1 \oplus \langle [A_i,A_j]\rangle$.
Let C be corresp Lie group.
Thm (Y.) C is Lipschitz 1-connected.
But G is not:

<u>Lifting</u>: A Lipschitz curve $[0,1] \to W_1$ lifts to a unique Lip curve in G or C.

A Lipschitz disc ~~$[0,1]^2$~~ $D^2 \to W_1$ lifts to G iff ~~$[Df(X_1), D$ in $Df$ is an ab~~ $[Df(X_1), Df(X_2)]=0$



$[\ ,\ ]_G$ restricts to 0 on im $Df_x$ $\forall x \in D^2$

Likewise for lifts to C.

But! $[v,w]_G = 0 \Leftrightarrow [v,w]_C = 0$. So every ~~horizontal~~ Lipschitz disc in G lifts to a Lipschitz disc in G.

And there's the problem: Let $\gamma:[0,1] \to G$ be a closed horizontal curve that ~~does~~ lifts to a path $\tilde\gamma$ in C.

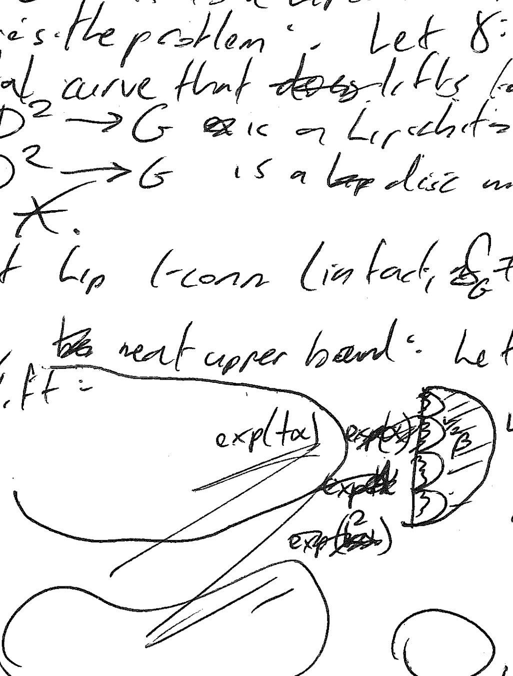
If $\beta: D^2 \to G$ is a Lipschitz ~~disc with $\partial\beta=\gamma$ then~~ with $\partial\beta = \gamma$

$\tilde\beta: D^2 \to G$ is a Lip disc with $\partial\tilde\beta = \tilde\gamma$ — but $\tilde\gamma$ isn't closed ✗.

So G is not Lip 1-conn (in fact, $\delta_G \neq n^2$)

Conversely, the upper bound: Let $\gamma$ be a closed curve in G, $\tilde\gamma$ it's lift:

Diff




← Area = Area(β) + 4 Area($s_{1/2}(\beta)$) + ...

$\exp\frac{t\alpha}{4} D$ = Area(β) + Area(β) + ... + $L^2$

$\lesssim L^2 \log L$.

$\gamma$ $L$ $s_{1/2}(\gamma)$