~~So $d_{f(p)}$~~ ~~In particular, if~~

$\partial_\infty X \cong \partial_\infty Y$.

So f_∂ is uniformly continuous?? ~~in fact a homeomorphism~~.

~~Explicitly So Conse~~ Therefore $\mathbb{H}^n \not\cong_{QI} \mathbb{H}^m$ unless $m=n$! — we can distinguish hyperbolic spaces.

~~Thm (Kapovich–Kleiner): If~~ Def: A finitely-generated group G is hyperbolic if ~~the~~ any Cayley graph of G is δ-hyperbolic for some δ.

~~Then any Ca~~ All Cayley graphs are geodesic, QI, so they all have homeomorphic ideal boundaries. (break?)

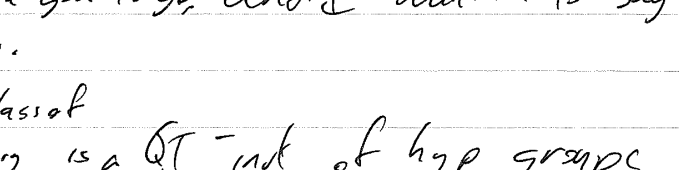
~~We~~ There's a geometry seminar tomorrow on conformal dimension, which ~~is related~~ was developed to ideal bdries — encourage you to go, and I wanted to say sthing brief about ~~some of the~~ conf dim.

Problem: We saw that ~~bdr~~ homeo. class of ideal boundary is a QI-inv't of hyp groups. But it turns out that ~~these~~ most hyp gps have the same boundary:

Thm (Kapovich–Kleiner): If X is a hyp. gp and ~~$\partial_\infty X$ has~~ $\dim \partial_\infty X = 1$, then $\partial_\infty X$ is homeo to:
- S^1
- The Sierpinski carpet
- the Menger curve.

S^1: example is $\mathbb{H}^2$.

Sierpinski: let M be a closed hyperbolic 3-manifold with totally geodesic boundary. Then $\tilde{M}$ is a subset of $\mathbb{H}^3$. In fact, it's $\mathbb{H}^3$ minus ∞'ly many half-spaces:



Menger: harder to describe: ~~The~~ like Sierpinski but not planar.

But Menger is generic: Thm (Dahmani–Guirardel–Przytycki)

How do you distinguish Menger spaces? Def: The conformal dimension of a space X is $\inf\{\dim_{Haus}(Y) \mid Y \text{ is quasisymmetric equiv to } X\}$. — This is QS invt.

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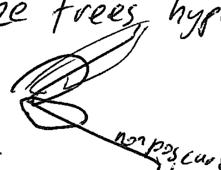
Growth.

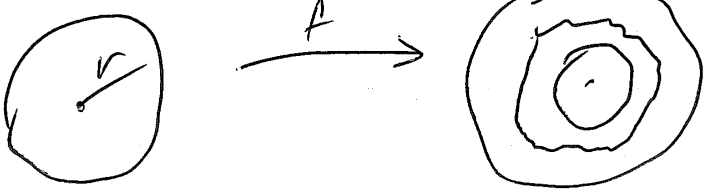
Previously, hyperbolic groups – generalize trees, hyperbolic space, negative curvature
In the map from the beginning of the class,



Now, I want to look at the other edge - amenable gps. specifically, groups of polynomial growth.

Def: Let G be a finitely generated group, $S \subset G$ a generating set. symmetric $S = S^{-1}$
The growth function $\gamma_S : \mathbb{N} \to \mathbb{N}$ is $\gamma_S(n) = \#B(n, d_S)$.
= number of elts of G that can be written as products of $\leq n$ elts of S.

Thm – This is at most exponential – $\gamma_S(n) \leq (2(\#S))^n$
– If $S, T \subset G$ two finite gen sets, then $\exists M$ s.t. $\gamma_T(\frac{n}{M}) \leq \gamma_S(n) \leq \gamma_T(Mn)$.
– If G, H are QI, S generates G, T generates H, then $\exists C$ s.t.
$$C^{-1}\gamma_T(C^{-1}n - C) \leq \gamma_S(n) \leq C\gamma_T(Cn + C)$$



Ex: Prove this.

– If $S \subset G$ is a gen. set and $\exists d \in \mathbb{R}$ s.t. $an^d \leq \gamma_S(n) \leq bn^d$ ($\exists a, b \in \mathbb{R}$), then $\forall H \sim_{QI} G$ and $\forall T \subset H$ gen sets $\gamma_T(n) \sim n^d$
(i.e. degree of polynomial growth is QI-invt.) We write $\gamma_G \sim n^d$
– If $S \subset G$ and $\exists c > 1 \in \mathbb{R}$ s.t. $\gamma_S(n) \geq c^n$ then $\forall H \sim_{QI} G$ and $\forall T \subset H$ $\exists c' > 1$ s.t. $\gamma_T(n) \geq (c')^n$. We write "G has exp. growth."
(exponential growth is QI-invt.)

Ex: $G = \mathbb{Z}^n$, $\gamma_G \sim n^n$ Ex: $G = F_n$ (free group of rank n), $n \geq 2$, γ_G is exponential,

Thm: If G is hyperbolic and $G \not\sim_{QI} \mathbb{Z}$, then γ_G is exponential.

~~Ex~~ Goal: Thm (Gromov): Let G be a fin. gen. gp. ~~If $\exists d \in \mathbb{R}$ st~~
~~$\gamma_G \simeq$~~ If $\exists$ ~~$d \in \mathbb{R}$~~ s.t. $\gamma(r) \leq p(r)$, then ~~$\exists$~~ $\exists\, d \in \mathbb{N}$ s.t. $\gamma_G \simeq r^d$
and G is ~~poly~~ virtually nilpotent ($\exists\, H \leq_{f.i.} G$ s.t. H is nilpotent).
Ex: nilpotent groups.

Recall: G is nilpotent if the lower central series terminates:
I.e, let $G_1 = [G, G]$

$G_{i+1} = [G, G_i]$ — then G is nilp $\Leftrightarrow \exists\, k$ s.t. $G_k = 0$.
If $G_k \neq 0$, $G_{k+1} = 0$, we say that G has step k.

Ex: Heisenberg group:
$$H = \langle X, Y, Z \mid [X, Y] = Z,\ [X, Z] = [Y, Z] = 1 \rangle.$$
- has a nice coordinate system: $\forall g \in H$, $\exists!\, a, b, c \in \mathbb{Z}$ st.
$$g = X^a Y^b Z^c.$$
- has a nice metric: ~~$\mathcal{L}$~~ let $S = \{X^{\pm 1}, Y^{\pm 1}, Z^{\pm 1}\}$, then
$$d_S(1, X^a Y^b Z^c) \sim |a| + |b| + \sqrt{|c|}.$$
(because $[X^n, Y^n] = Z^{n^2}$)
- has ~~grow~~ $\gamma_H \sim r^4$.

~~Thm (Bass-Guivarch)~~: In general: ~~torsion~~
Thm (Malcev, ~~Bass-Guivarch~~): Any f.g. nilpotent group G:
- has a finite-index torsion-free subgroup.
- if G is torsion free, it embeds in a nilpotent Lie group N.
where $\dim N = \sum \operatorname{rank}(G_k / G_{k+1})$

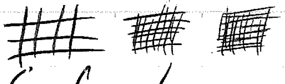
Thm (~~Bass~~-Guivarch): $\gamma_G = r^Q$, where $Q = \sum k \operatorname{rank}(G_k / G_{k+1})$

~~Thm~~ (Grigorchuk): First example of a group with intermediate growth:
~~large~~ $\gamma_G > r^d\ \forall d \in \mathbb{N}$, but $\gamma_G < c^r\ \forall c > 1$.

Thm (Gromov): ...

Goal: sketch proof. (break)

Asymptotic cones.

Main tool of Gromov's theorem: asymptotic cones   Ex: 

~~Asy~~ Idea: view a space from only far away. Can formalize using Gromov-Hausdorff

Recall: If X is a metric space, $A, B \subset X$ compact, the Hausdorff distance $d_H(A,B) = \min\{ r \mid B \subset N_r(A) \text{ and } A \subset N_r(B)\}$.

Def: Let X, Y be compact metric spaces. The Gromov-Hausdorff distance from ~~A to B~~ X to Y is $d_{GH}(X,Y) = \inf_M d_H(X,Y;M)$, where M ranges over metric spaces containing X, Y isometrically.

~~Ex: Ex: Lemma:~~

a ~~2D-almost isometry~~ $(1,D)$-QI

Lemma: Let $D = d_{GH}(X,Y)$. Then $\exists\ \varphi: X \to Y$ s.t. $\forall x, x' \in X$,

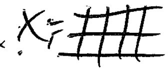
$$|d(\varphi(x), \varphi(x')) - d(x,x')| \leq 2D.$$

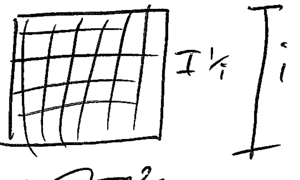
Conversely, if $\exists$ a $(1,D)$-QI from X to Y, then $d_{GH}(X,Y) \leq 2D$

~~A D-almost isom is a map $X \to Y$ s.t. ... (a $(1,D)$-QI). We say~~ S-let.

~~So we can do If $(X_1, \dots, X_n$ is a seq of metric spaces,~~

If $X_1, \dots, X_n$, is a seq. of metric spaces and Y is a metric space, we say that Y is a pointed G-H limit of (X_i) if $\exists$ $x_i \in X_i$, ~~st~~ $r_i > 0$, $y \in Y$ s.t.: $r_i \to \infty$ and $\lim d_{GH}(B(r_i, x_i; X), B(r_i, y; Y))$

Ex: $X_i =$  $\frac{1}{i}$, $X_i \xrightarrow{G-H} (\mathbb{R}^2, \ell_1)$ ~~likewise, $(\mathbb{Z}^2, \frac{1}{i} d_S) \xrightarrow{GH} (\mathbb{R}^2, \ell_1)$~~

Ex: $X_i =$  $\frac{1}{i}$ $X_i \xrightarrow{G-H} (\mathbb{R}^2, \ell_1)$.

(X,d) metric space,

~~Ex:~~ ~~Ex:~~ In particular, if $X_i = (X, \frac{1}{i} d)$, $X_i \to Y$, we say Y is a scaling limit of the X_i.

Sometimes easy: $(\mathbb{Z}^2, \frac{1}{i} d_S) \to (\mathbb{R}^2, \ell_1)$.

This is a lattice in a 3-dim Lie group

Sometimes harder: H = Heisenberg group. $(H, \frac{1}{i} d_S) \to \{\mathbb{R}^3$ with ~~sub R~~ subFinsler metric.

You can scale, but the center has to scale faster. result:

Idea: $x^i y^i x^{-i} y^{-i}$ is a path in $(H, \frac{1}{i} d_S)$ of length 4.

But ~~x^i~~ z^{i^2} is a path of length $i \to \infty$. So, some directions in the limit have infinite length.

But also, some don't converge: $T =$  doesn't converge in GH. Need a more powerful notion of convergence: ultralimits of metric spaces.

Ultralimits: Thm: $\exists$ a linear functional $L: l_\infty \to \mathbb{R}$ s.t. $\forall (x_i) \in l_\infty$, $L((x_i))$ is a point of accumulation of (x_i). ~~In particular~~ Write $\overline{\lim}\, x_i$.
~~Pf. uses axiom of choice (ultrafilters)~~ Then: $\overline{\lim}\, x_i \in [\inf x_i, \sup x_i]$. $\overline{\lim}\, x_i = \overline{\lim}\, y_i$ if $x_i = y_i$ except finitely many; $\overline{\lim}\, x_i \le \overline{\lim}\, y_i$ if $x_i \le y_i$ except finitely many.
~~(This is a little bit more than Hahn-~~ (Hahn-Banach lets you construct L s.t. $L((x_i)) = \lim x_i$ if it exists ~~a this takes a little more~~
— that's enough for most purposes, but sometimes we need a little more)
~~We'll write $\overline{\lim}$~~ We'll write $\overline{\lim}\, x_i = L((x_i))$

For a seq. of metric spaces, (X_i) and basepoints $x_i \in X_i$, we define
~~$X_\omega = (l_\infty(X)$~~ $l_\infty(X_i) = \{ \cancel{(a_i)}\ i \mapsto a_i \in X_i \mid \sup_i d(x_i, a_i) < \infty \}$.

~~$X_\omega = (l_\infty(X_i) / \sim, d$~~ For $(a_i), (b_i) \in l_\infty(X_i)$, let
$d_\omega((a_i), b_i)) = \cancel{L(d(a_i, b_i))}\ \overline{\lim}\, d(a_i, b_i)$.

~~Then d is a pseudometric (satisfies Δ ineq.)~~
~~Then $d((a_i), (b_i))$~~ Then d is satisfies Δ-ineq (pseudometric).

~~Let X_ω~~ The ultralimit of (X_i, x_i) is $X_\omega = \left(\frac{l_\infty(X_i)}{\sim}, d_\omega \right)$
where $(a_i) \sim (b_i) \iff d_\omega((a_i), (b_i)) = 0$. This is a lot to unpack, so let's try a simple example:

Ex: $X_i = \mathbb{R}$.

Ultralimits:

Def: ~~Rea~~ A <u>filter</u> is a subset $\omega \subset 2^{\mathbb{N}}$ s.t. (equiv. a fn $\omega : 2^{\mathbb{N}} \to \{0,1\}$)

- ω is closed under finite intersections
- If $A \in \omega$ and $B \supset A$, then $B \in \omega$ (some sets are big, some small)
- $\emptyset \notin \omega$ and $\mathbb{N} \in \omega$.

ω is an <u>ultrafilter</u> if $\forall A \subset \mathbb{N}$, either $A \in \omega$ or $\mathbb{N} \setminus A \in \omega$.

~~Equiv. ω is a finitely additive measure on $\mathbb{N}$ s.t. every set every set which takes vals in $\{0,1\}$~~

Lemma: ω is a finitely additive measure on $\mathbb{N}$, i.e. if $A \cap B = \emptyset$, then $\omega(A \cup B) = \omega(A) + \omega(B)$.

Pf:

$\omega(A)$	$\omega(B)$	$\omega(A\cup B)$	
1	1	impossible:	$A \cap B = \emptyset \notin \omega$.
1	0	1	$A \cup B \supset A$
0	1	1	$A \cup B \supset B$
0	0	0	$(\mathbb{N}\setminus A) \cap (\mathbb{N} \setminus B) = \mathbb{N} \setminus (A \cup B) \in \omega \Rightarrow A \cup B \notin \omega$. //

~~Observe~~ Easy example: A principal ultrafilter is a filter
$\omega_a = \{A \subset \mathbb{N} \mid a \in A\}$. (point measure).

Prop: $\exists$ a nonprincipal ultrafilter. Pf: Axiom of Choice.

Using this ultrafilter, can define <u>ultralimits</u>. If ω nonprincipal, then ω contains no finite sets

Thm: Let ω be a nonprincipal ultrafilter. Let $(x_i) \in \ell_\infty$, $y \in \mathbb{R}$.
We say $\lim_\omega x_i = y$ if $\forall \varepsilon > 0$,
$\{i \in \mathbb{N} \mid |x_i - y| < \varepsilon\} \in \omega$.
Then every $(x_i) \in \ell_\infty$ has a unique $\lim_\omega x_i$.

Pf: ~~WLOG~~ Suppose $\{i \in \mathbb{N} \mid x_i \in [a,b]\} \in \omega$. Let $m = \frac{a+b}{2}$.
Then either $\{i \in \mathbb{N} \mid x_i \in [a,m]\} \in \omega$ or
$\{i \in \mathbb{N} \mid x_i \in (m,b]\} \in \omega$.
So we can find $\lim_\omega x_i$ by ~~bis~~ bisection.
Further, $\lim_\omega x_i$ is unique:

This is like a limit, except it exists for every bdd sequence.
Thm: $\lim_\omega$ is additive, ~~monotone, etc.~~ — if $x_i = y_i$ except for fin. many i, then $\lim x_i = \lim y_i$, etc.