Boundaries and Quasi-isometries

Last time: Ideal boundary = $\{$geodesic rays$\}/\sim$ where $\gamma_1 \sim \gamma_2$ if they stay bdd dist apart. ~~Then~~, Particularly suited to QIs: by Morse, if $f: X \to Y$ is a QI, γ a ray, then $f(\gamma)$ is close to a ray $\Rightarrow$ f induces $\partial_\infty f: \partial_\infty X \to \partial_\infty Y$.

Ex: $\partial_\infty \mathbb{H}^n = S^{n-1}$. $\mathrm{Isom}\, \mathbb{H}^n$ acts on S^{n-1} by conformal maps.

But this was mostly an extrinsic argument — we know $\mathrm{Isom}(\mathbb{H}^n)$, we can show that ~~if~~ each isom acts by a conformal map.

Can we argue intrinsically? Can we extend to QI's?

Start with a theorem: Let $f: \mathbb{H}^n \to \mathbb{H}^n$ be an (L,C) QI. Then $\partial f: S^{n-1} \to S^{n-1}$ is a quasi-conformal map, with QC constants depending on ~~the QI~~ L and C. Need a def and a lemma.

Def: A conformal map ~~sends~~ "preserves angles". Equiv, if $g: \mathbb{R}^n \to \mathbb{R}^n$ ~~then~~ and g is smooth, then g is conformal $\Leftrightarrow D_x g = \lambda(x) I_n$ $\forall x \in \mathbb{R}^n$. ($g$ sends infinitesimal circles to infinitesimal circles.)

We can also define w/o smoothness: for $x \in \mathbb{R}^n$ let

$$H_x(g) = \limsup_{\rho \to 0} \max_{\substack{|y-x|=\rho \\ |z-x|=\rho}} \frac{|f(y)-f(x)|}{|f(z)-f(x)|}.$$

If ~~$H_x(g)$~~ g is conformal, then ~~(Nontrivial to show th~~ $H_x(g) = 1$ $\forall x \in \mathbb{R}^n$. ~~(Converse is true, more~~ (Converse is true, but nontrivial ~~Gehring~~) ~~in $n \geq 3$~~.

If $H_x(g) < C$, we say that g is <u>C-quasiconformal</u>. Gehring: g is diff a.e., so: (~~g sends~~ Roughly, g sends infinitesimal circles to ellipsoids with bdd eccentricity: $\circ \to 0$)

~~Pf: Pf: Pf: QC maps Conformal maps are QC and compositions~~

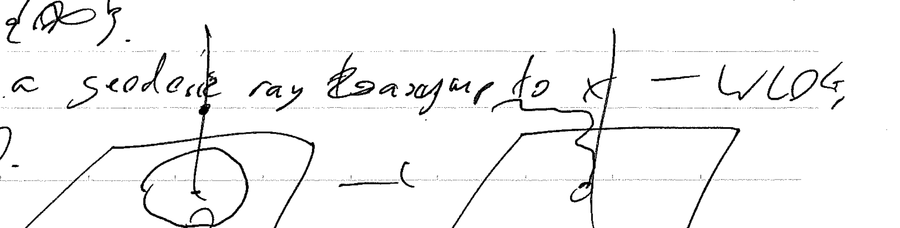
~~Thm (Heinonen)~~: Pf: Conformal maps are QC and compositions of QC maps are QC (nontrivial), ~~so it suffices to show that~~ so we can ~~repla~~ work in upper half-space model.

i.e. $\partial_\infty X = \mathbb{R}^{n-1} \cup \{\infty\}$.

Let $x \in \mathbb{R}^n$, let γ be a vertical geodesic ray ~~asymp~~ to x — WLOG, take $x = 0$, $f(x) = 0$.

In fact, $f_*(\gamma)$ close to vertical

What happens to circles?



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Boundaries and QIs

Last time: $\partial_\infty X = \{\text{geodesic rays}\}/\sim$ where $\gamma_1 \sim \gamma_2$ if γ_1, γ_2 stay ~~bdd~~ bdd dist apart. Particularly useful for QIs of hyp spaces:

Prop: Let $f: X \to Y$ be a QI between δ-hyp geod metric spaces. Then f induces a bijection $f_\infty: \partial_\infty X \to \partial_\infty Y$.

Pf: Namely, for $[\gamma] \in \partial_\infty X$, ~~let $f_\infty([\gamma])$~~ $f(\gamma)$ is ~~bdd~~ a QG, so it's bdd dist from some ray λ — let $f_\infty([\gamma]) = [\lambda]$.
Show: well-defined, injective, surjective //

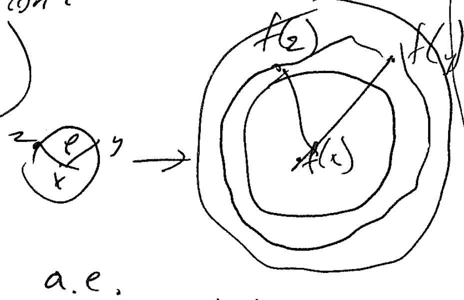
But we really want more: $\mathrm{Isom}(\mathbb{H}^n)$ acts on $\partial_\infty \mathbb{H}^n \cong S^{n-1}$ by conformal maps. Can we generalize to δ-hyp spaces?

Yes! Need some defs and facts:

Def: for smooth maps $f: \mathbb{R}^n \to \mathbb{R}^n$, f is <u>conformal</u> if "preserves angles" — i.e. ~~$\exists \lambda$ s.t. $\forall x \in \mathbb{R}^n$, $D_x f = \lambda(x)$~~ $\forall x \in \mathbb{R}^n$, $D_x f$ is rotation + scaling.

We'd like to define this for nonsmooth maps, for maps of metric spaces.
~~So Main idea:~~ Main idea: conformal map sends infinitesimal ~~circles~~ spheres to ~~circles~~ spheres ~~too~~

~~So~~ One way: Quasiconformal map:
- ~~if f is smooth, f is QC if $D_x f$ has bounded dilatation $\Rightarrow$ sends circle to ellipsoids (ratio of singular values of $D_x f$ is bdd)~~ ~~spheres~~
- ~~if f is not~~ ~~in general~~, def dilatation function:
$$H_x(f) = \limsup_{\rho \to 0} \left(\frac{\max_{|y-x|=\rho} |f(x)-f(y)|}{\min_{|z-x|=\rho} |f(x)-f(z)|} \right)$$
f is C-QC if $H_x(f)$ is bdd by C.



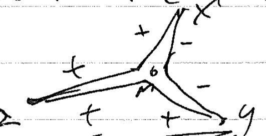
Thm (Gehring): if $f: \mathbb{R}^n \to \mathbb{R}^n$ is QC, then $D_x f$ exists a.e. and the ratio of any two singular values of $D_x f$ is bdd.
In particular, if f is 1-QC, it is conformal. (and conformal $\Rightarrow$ smooth)

This def works best in $\mathbb{R}^n$ — if we generalize ~~to~~ to metric spaces, use:

Def: $f: X \to Y$ is <u>quasi-symmetric</u> if ~~$\forall x, y, z \in X$~~, $\exists \eta: \mathbb{R}^+ \to \mathbb{R}^+$, s.t. $\forall x, y, z$
$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \le \eta\left(\frac{|x-y|}{|x-z|}\right).$$
(When $f: \mathbb{R}^n \to \mathbb{R}^n$, QS $\Leftrightarrow$ QC.)

First, need a metric on $\partial_\infty X$. Can define based on Gromov product.

Def: Let $x,y,z \in X$. Define $(x,y)_z = \frac{1}{2}(d(z,x)+d(z,y)-d(y,x))$
(Can define this anywhere — esp usefull in hyp spaces, because:



$(x,y)_z$ is approx $d(z,m)$, where ~~m~~ m is center of Δ
— measures how long $[z,x]$ and $[z,y]$ stay close.
~~For any X,~~ When X is a tree, this is exact: $(x,y)_z$ is how long $\overline{xz}, \overline{yz}$ overlap.

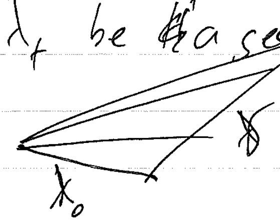
~~Ex:~~ Ex: If $\lambda, \gamma : [0,\infty) \to X$ are geodesic rays, $\lambda(0)=\gamma(0)=p$,
then $(\lambda(t),\gamma(t))_p$ is nondecreasing, ~~so we can extend~~
~~Ex: If X is δ-hyp, then $(\lambda(t),\gamma(t))_p$ is coarsely nonincreasing:~~
~~∃C s.t. $(\lambda(t),\gamma(t))_p$ so we~~ , so we can define
$(\lambda,\gamma)_p = \lim_{t\to\infty} (\lambda(t),\gamma(t))_p$. If $\gamma \sim \lambda$, then
$(\lambda(t),\gamma(t))_p = \frac{1}{2}(t+t-d(\gamma(t),\lambda(t)) \to \infty$.

Ex: if X is δ-hyp and $\gamma \not\sim \lambda$, then
$(\lambda,\gamma)_p < \infty$. Further, if $\lambda \sim \lambda'$, $\gamma \sim \gamma'$
then $|(\lambda',\gamma')_p - (\lambda,\gamma)_p| < 4\delta$.

This lets us define a visual metric on $\partial_\infty X$. Suppose X is proper, geodesic, δ-hyp.

Lemma: Let $x \in X$, $[\gamma] \in \partial_\infty X$. Then $\exists$ a geodesic ray λ s.t. $\lambda \sim \gamma$ and $\lambda(0)=x$.

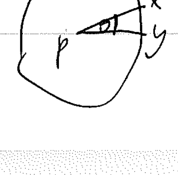
Pf: ~~Consider the~~ For each t, ~~consider the geodesic segment~~ $\overline{x\gamma(t)}$:
~~$\lambda(t)$ let $\lambda_t : [0,t] \to X$, be a geod. s.t. $\lambda_t(0)=x$, $\lambda_t($~~
let λ_t be a geodesic from x to $\gamma(t)$.



By δ-hyp, λ_t is δ-close to $\lambda_0 \cup \gamma$.
So by Arzéla-Ascoli, some subseq. of λ_t converges to the desired λ. //

We say that a metric d on $\partial_\infty X$ is a visual metric if $\exists p \in X$, $\exists A, B > 0$, $c > 1$, s.t. $\forall$ rays λ, γ with $\lambda(0)=\gamma(0)=p$,
$$B c^{-(\lambda,\gamma)_p} \le d([\lambda],[\gamma]) \le A c^{-(\lambda,\gamma)_p}$$

Ex: The round metric on S^{n-1} is a visual metric for $\mathbb{H}^n$:



~~arc has length~~ $(x,y)_p = \log \csc(\frac{\theta}{2}) \approx -\log \frac{\theta}{2}$ when θ is small.

idea: arc of radius r has length $\approx e^r \cdot \theta$ — ~~so length~~ exponential growth with ~~dist~~ ~~where~~ when $r \approx -\log\theta$. — split at $r \approx -\log\theta$.

S, $\angle_p(x,y) \approx e^{-(x,y)_p}$.

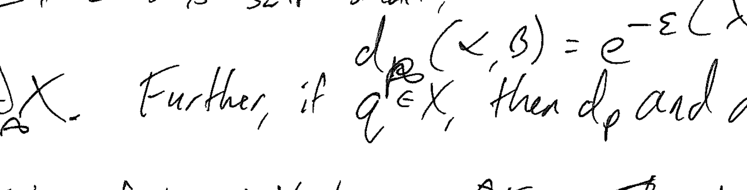
Ex: $X = T$  $\partial_\infty X = \{$infinite 0-1-sequences$\}$,
binary tree. If $\alpha, \beta \in \partial_\infty X$, then
$$(\alpha,\beta)_p = \sup\{n \mid a_i = b_i \text{ for } i = 1, \dots, n\}.$$

Let $d_\infty(\alpha,\beta) = e^{-(\alpha,\beta)_p}$ — ~~this is an ultrametric on $\partial_\infty X$ = Cantor~~
~~on~~ this gives $\partial_\infty X$ ~~the topology of~~ homeo to a Cantor set.

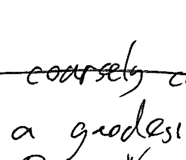
Thm: Let X be a geodesic, proper, δ-hyp space. ~~Then~~ ~~If $\varepsilon > 0$ is suff small,~~ ~~then d_∞~~ For each $\alpha \in \partial_\infty X$, choose a rep λ_α.
If $\varepsilon > 0$ is suff small, then
$$d_p(\alpha,\beta) = e^{-\varepsilon(\lambda_\alpha, \lambda_\beta)_p}$$ is a visual metric
on $\partial_\infty X$. Further, if $q \in X$, then d_p and d_q are bilipschitz equivalent.

Thm: Let $f : X \to Y$ be a QI. ~~Then~~ Let $p \in X$ and let d_p, $d_{f(p)}$
be metrics as above. Then f_∞ is a homeomorphism from $\partial_\infty X$ to $\partial_\infty Y$.

~~Pf:~~ Lemma: ~~QI's coarsely commut~~
Let γ be a geodesic ray. Let $P_\gamma : X \to \gamma$ be nearest-point projection
i.e. $P_\gamma(x) \in \gamma$ minimizes $d(x, P_\gamma(x))$. Let $f : X \to Y$
and let $\lambda \in Y$ a geo ray s.t. $d(\lambda, f(\gamma)) < \infty$. Then $\exists C$ s.t.
$$d(f(P_\gamma(x)), P_\lambda(f(x))) < C \quad \forall x \in X.$$
(Projection commutes with QIs).

Pf:  Exercise: Fill in the details. //

Cor: Let γ, λ be geodesic rays, let $\overline{f(\gamma)}, \overline{f(\lambda)}$ (based at p) be rays (based at $f(p)$) asymp to $f(\gamma), f(\lambda)$.
Then $$C^{-1}(\gamma,\lambda)_p - C \le (\overline{f(\gamma)}, \overline{f(\lambda)})_{f(p)} \le C(\gamma,\lambda)_p + C$$

Pf: $(\gamma,\lambda)_p \approx d(p, P_\lambda(\gamma(t)))$ when t is large. 
$$(\overline{f(\gamma)}, \overline{f(\lambda)})_{f(p)} \approx d(f(p), P_{\overline{f(\lambda)}}(\overline{f(\gamma)}(t))) \approx d(f(p), f(P_\lambda(\gamma(t)))) //$$