No.
Date. 2024 04 02

Morse Index Thm II

Last time: $\gamma_\tau = \gamma([0,\tau])$, $H_\tau = H(E)_{\gamma_\tau}$, $\lambda(\tau) = \text{index } H_\tau$.
$J_\tau = \{\text{broken Jacobis}\}$ s.t. $\dim J_\tau < \infty$, $\text{index } H_\tau|_{J_\tau} = \lambda(\tau)$.

① $\lambda(\tau)$ is nondecreasing
② if $n = \text{nullity}(H_\tau) = \dim\{W \mid W \text{ is Jacobi}, W(0)=0, W(\tau)=0\}$
$= \text{ord}_{\gamma(0)} \gamma(\tau)$

then $\forall \tau$ $\exists \delta > 0$ s.t. if $|h| < \delta$, then $\lambda(\tau+h) \in [\lambda(\tau), \lambda(\tau)+n]$.

Pf: Decompose $J_\tau = V_- \oplus V_0 \oplus V_+$ - if $|h|$ small then
$H_{\tau+h}|_{V_-} < 0$, $H_{\tau+h}|_{V_+} > 0$, just leaving V_0.

Further, if $0 < h < \delta$, then $\lambda(\tau - h) = \lambda(\tau)$ (nondecreasing)
$\lambda(\tau+h) \leq \lambda(\tau) + n$

Need to show: $\lambda(\tau+h) \geq \lambda(\tau)+n$. - Find $\lambda(\tau)+n$-dim subsp.

Pf: Let $k = \text{index}(H_\tau)$, let $W_1, \dots, W_k \in T_{\gamma_\tau}\Omega$ generate a neg-def subspace.
Let $J_1, \dots, J_n \in T_{\gamma_\tau}\Omega$ linearly indep. Jacobi fields.
Extend on $[\tau, \tau+h]$ by 0 to get $\overline{W}_i, \overline{J}_j$.
Claim: $S = \langle \overline{W}_1, \dots, \overline{W}_k, \overline{J}_1, \dots, \overline{J}_n \rangle$ can be perturbed to be neg. def. We have $J_i \in \text{null}(H_\tau)$, so $H(\overline{J}_i, \overline{W}_j) = 0$
$H(\overline{J}_i, \overline{J}_j) = 0$

As a matrix,
$$H|_S = \begin{array}{c|cc} & \overline{W}_1 \cdots \overline{W}_k & \overline{J}_1 \cdots \overline{J}_n \\ \hline \overline{W} & M & 0 \\ \overline{J} & 0 & 0 \end{array}$$
where M is negative definite (symmetric, all eigvals < 0).

Need to perturb so bottom right is also neg def.
Let $z_i = \Delta D_t J_i(\tau)$. Then for $Y \in T_{\gamma_{\tau+h}}\Omega$, $H(\overline{J}_i, Y) = -\langle z_i | Y(\tau)\rangle$
Since z_i are lin. indep, $\exists$ a dual basis y_i s.t. $\langle y_i | z_j \rangle = \delta_{ij}$.
Let $Y_i \in T_{\gamma_{\tau+h}}\Omega$ fields s.t. $Y_i(\tau) = y_i$. $\Rightarrow H(\overline{J}_i, Y_j) = -\delta_{ij}$

Let $c>0$, let $S' = \langle \overline{W}_1, \dots, \overline{W}_k, c^{-1}\overline{J}_1 + cY_1, \dots, c^{-1}\overline{J}_n + cY_n \rangle$ - then

$$H|_{S'} = \left(\begin{array}{c|c} M & cH(\overline{W}_i, Y_j) \\ \hline " & -2I + c^2 H(Y_i, Y_j) \end{array} \right)$$

$H(c^{-1}\overline{J}_i + cY_i, c^{-1}\overline{J}_j + cY_j)$
$= H(\overline{J}_i, Y_j) + H(\overline{J}_j, Y_i) + c^2 H(Y_i, Y_j)$
$= -2\delta_{ij} + c^2 H(Y_i, Y_j)$

When $c \to 0$, $H|_{S'} \to \left(\begin{array}{c|c} M & 0 \\ \hline 0 & -2I \end{array}\right) < 0$. //

App: Let M be a mfld, with nonpositive sectional curvature: i.e. $K(X,Y) \leq 0$ $\forall X,Y$ linearly indep.
Then ~~index~~ $(H_\gamma) =$ ~~0~~ nullity$(H_\gamma) = 0$ $\forall$ geodesic γ.
Pf: ETS no conjugate pts: Let γ a geodesic, W a Jacobi field, consider $\langle D_t W | W \rangle$ we have

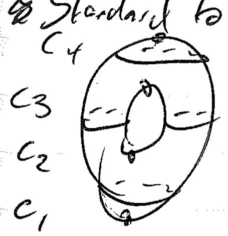
$$\begin{aligned}\frac{d}{dt}\langle D_t W | W\rangle &= \|D_t W\|^2 + \langle D_t^2 W | W\rangle \\ &= \|D_t W\|^2 + \langle R(\gamma', W)\gamma', W\rangle \\ &= \|D_t W\|^2 - K(\gamma', W)\underbrace{(\|\gamma'\|^2\|W\|^2 - \langle\gamma' | W\rangle^2)}_{\geq 0} \\ &\geq 0.\end{aligned}$$

So $\langle D_t W | W\rangle$ is nondecreasing. If $W(0)=0$, $W(T)=0$, then $\langle D_t W | W \rangle = 0$ $\forall t \in [0,1] \Rightarrow \|D_t W\|^2 = 0 \Rightarrow W(t) = 0$ $\forall t \in [0,T]$

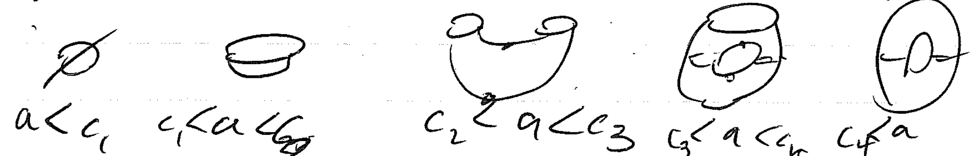
~~Almost says that every geodesic is locally length-m~~ ~~infinitesimal~~
Almost says that if γ is a geod, then every suff. close curve is longer, but not quite. In fact, we can get that, plus more, but we need topology. Two main ideas:
① Approximate Ω by a manifold s.t. $E|_\Omega$ is smooth, every geod is a crit pt with index = index(H) ~~and $E|_\Omega$~~
② If we know the crit pts of $E|_\Omega$, what can we say about Ω?

Morse theory: Let M a manifold, let $f: M \to \mathbb{R}$ a smooth function. A crit pt of f is a pt $p \in M$ s.t. $\nabla f(p) = 0$. We say p is nondegenerate if null$(H(f)_p) = 0$ and write index(p) = index $H(f)_p$

Standard to draw like so: height function on torus:



Four crit pts index 0, 1, 1, 2.
Let $M^a = f^{-1}((-\infty, a])$. Then idea of Morse theory is ~~that we can study~~ to study M^a as a increases:



- M^a only changes topology at crit pts
- the type of change depends on index(p).

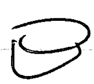
First part isn't too hard:

Prop: Let $a < b$ and suppose $f^{-1}[a,b]$ is compact, contains no crit pts of f. Then M^a is diffeo to M^b ~~For fact~~, and M^a is a deformation retract of M^b (Gradient flow)

Pf: Let $W = \frac{-\nabla f}{\|\nabla f\|^2}$. This is smooth on $f^{-1}([a,b])$ and $Wf = -1$.
Thus, for any flow line γ of W, $\frac{d}{dt} f(\gamma(t)) = -1$.

Let $\phi(x) = \begin{cases} x & \text{if } x \in M^a \\ \text{flow } x \text{ along } W \text{ for time } f(x)-a & \text{if } x \notin M^a \end{cases}$

This is a deformation retraction of M^b to M^a, can smooth to get a diffeo. //

Second is harder: what happens when we go from  to  ?

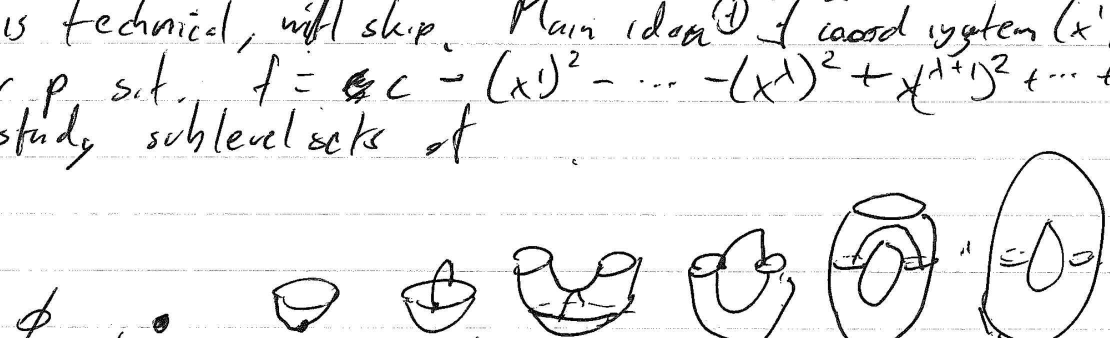
~~Really,~~  ~~to~~  ~~glue those~~ together. Can't retract anymore, but if we add a bridge, can retract!

Prop: Let $c \in \mathbb{R}$, $\varepsilon > 0$, suppose $f^{-1}([c-\varepsilon, c+\varepsilon])$ is compact, contains exactly one crit pt p, ~~with f(p)= which is nondegenerate,~~ ~~with f(p)=c~~. ~~f(p)=c~~. Further, $f(p) = c$, ~~index(p)=λ~~, p is nondegenerate & $\mathrm{index}(p) = \lambda$. Then $M^{c+\varepsilon}$ deformation retracts to $M^{c-\varepsilon}$ with a λ-cell attached.

That is there is ~~a map~~ an attaching map $\alpha : \partial D^\lambda = S^{\lambda-1} \to M^{c-\varepsilon}$ s.t. $M^{c+\varepsilon}$ ~~retracts~~ to $M^{c-\varepsilon} \cup_\alpha D^\lambda = \dfrac{M^{c-\varepsilon} \sqcup D^\lambda}{p \sim \alpha(p)\ \forall p \in S^{\lambda-1}}$

Pf is technical, will skip. Main idea ① $\exists$ coord system $(x^1, \dots, x^n)$ (Morse lemma) near p s.t. $f = c - (x^1)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2$
② study sublevel sets of

Ex:



~~glue~~ attach 0-cell (point) — attach 1-cell (line) — attach 1-cell — attach 2-cell

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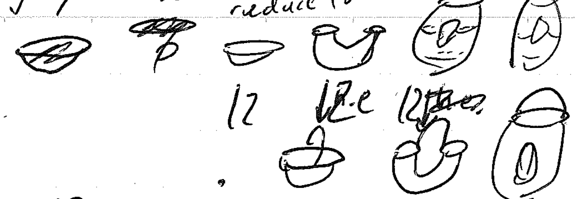
Morse theory / Story of algebraic topology:
- We want to know whether A and B are homeo.
- Define invariants i.e. fund gp, higher homotopy groups, reduce to algebra.
- Problem: ~~these~~ two spaces can be homotopically indist
- this is a homotopy equivalence

Last time:  ~~thus~~ How does M_a change? h.e. to gluing on a cell.

h.e. h.e.

Prop: Let $c \in \mathbb{R}$, $\varepsilon > 0$, suppose $f^{-1}([a,b])$ is compact, contains exactly one critical point p. Further, $f(p) = c$, p is nondegenerate, $\mathrm{index}(p) = \lambda$. Then $M_{c+\varepsilon}^{c+\varepsilon}$ deformation retracts to $M_{c-\varepsilon}^{c-\varepsilon}$ with ~~a~~ λ-cell attached.

That is, there is an attaching map $\alpha : \partial D^\lambda = S^{\lambda-1} \longrightarrow M_{\varepsilon-\varepsilon}^{\varepsilon-\varepsilon}$ s.t. $M_{c+\varepsilon}^{c+\varepsilon}$ retracts to $M_{c-\varepsilon}^{c-\varepsilon} \cup_\alpha D^\lambda = M^{c-\varepsilon} \amalg D^\lambda \,/\, p \sim \alpha(p)\ \forall p \in \partial D^\lambda$

$\lambda = 0$ - add a point, $\lambda = 1$ - connect two pts by an edge

Pf ~~depends on~~ Pf is technical, will skip, but it depends on the index λ of ~~f(p)~~:

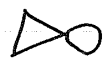
Morse Lemma: Let p be a nondegenerate crit pt of f. Then $\exists$ a coord chart $(x^1, \dots, x^n) : U \longrightarrow \mathbb{R}^n$ s.t.

$$f(x^1, \dots, x^n) = f(p) - (x^1)^2 - \dots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \dots + (x^n)^2.$$

- so it's enough to consider ~~the case of~~ the case that f is a quadratic form.

~~So: Now let's~~ To put this together, we need one more ~~concept~~ ingredient.

Def: A CW complex is a space constructed by inductively gluing cells:

$X^{(0)} =$  $X^{(1)} =$  $X^{(2)} =$  etc.

These have nice inductive properties:

Thm (Whitehead, Hilton): Let X be a topological space which is h.e. to a CW complex Y. Let $\alpha : S^{\lambda-1} \to X$. Then $X \cup_\alpha D^\lambda \simeq Y \cup_\beta D^\lambda$ for some map $\beta : D^\lambda \to Y^{(\lambda-1)}$, i.e. $X \cup_\alpha D^\lambda$ is h.e. to a CW cplex with one more cell. So:

~~Thm induct the~~ Thm (Morse): If $f : M \to \mathbb{R}$ is a Morse function ($f^{-1}((-\infty, a])$ compact for all a, all crit pts nondegenerate) then M is htpy equiv to a CW complex with one λ-cell for each critical pt of index λ.

Remarkably powerful: ~~you may~~ have heard ~~about exotic spheres~~ -

~~M is a smooth manifold s.t. M is homeomorphic to~~

Ex: Let M be a compact n-manifold. If $f: M \to \mathbb{R}$ is a Morse function with exactly two critical pts, then $M \simeq S^n$

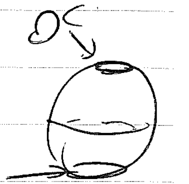
Pf: Since M is compact, one crit pt is a minimum (index 0) one is a max (index n). So $M \simeq X$ with one 0-cell, one n-cell:

$$X^{(0)} = \{p\}, \quad X = X^{(n)} = \{p\} \cup_\alpha D^n, \text{ where } \alpha: S^{n-1} \to \{p\}$$

$\Rightarrow \alpha(x) = p \ \forall x \Rightarrow X = D^n/_{S^{n-1}} = S^n$ //

In fact, M is homeomorphic (but poss. not diffeo) to S^n.

Pf: Let $p = \min$, $q = \max$, $f(p) = a$, $f(q) = b$.



By Morse Lemma, $\exists \varepsilon > 0$ s.t. $f^{-1}([a, a+\varepsilon]) \cong D^n = C_1$

$f^{-1}([b, b-\varepsilon]) \cong D^n$.

$f^{-1}([a+\varepsilon, b-\varepsilon])$ has no crit pts, so $M^{a+\varepsilon} \cong M^{b-\varepsilon} \cong D^n$

~~So $M \cong f^{-1}([b, b-\varepsilon]) \cup f^{-1}([a, b-\varepsilon])$~~ $f^{-1}([a, b-\varepsilon]) \cup f^{-1}([b-\varepsilon, b])$

So $\partial C_1 \cong S^{n-1}$ splits M into two discs — we can write ~~$\partial D^n \cup \partial D^n$~~ ~~$D^n \amalg D^n$~~

~~$M = D^n \cup D^n \; C_1 \cup_\alpha C_2$ where $\alpha: \partial C$~~

$M \cong D^n \cup_\alpha D^n$ where $\alpha: S^{n-1} \to S^{n-1}$ is a diffeo

Thm (Alexander): Any such mfld is homeo to S^n.

Pf: Let (r, θ), $(r, \theta)'$ be polar coords on C, C' resp. ~~Then~~ s.t. $(1, \theta) \sim (1, \alpha(\theta))'$. Give S^n polar coords $(r, \theta)^N, (r, \theta)^S$

Let $\beta: D^n \cup_\alpha D^n \to S^n$

$$(r, \theta) \longmapsto (r, \alpha(\theta))^N$$
$$(r, \theta)' \longmapsto (r, \theta)^S.$$

This is cts, not smooth at north pole

(In fact, this is how Milnor discovered the first exotic smooth structure on S^7!)

We'll use for the path space: ~~Recall~~

Let $p, q \in M$. $\Omega = \Omega(p, q) = \{$piecewise-smooth paths from p to $q\}$

For $w_1, w_2 \in \Omega$, let $d(w_1, w_2) = \max[d(w_1(t), w_2(t))] + \int_0^1 (\|w_1'(t) - w_2'(t)\|^2)$

Then: $E: \Omega \to \mathbb{R}$ is cts.

Not a manifold, but it has subsets that are mflds

Let $\Omega^a = E^{-1}([0,a])$, $\mathrm{Int}\,\Omega^a = E^{-1}([0,a))$

For $0 = t_0 < t_1 < \cdots < t_k = 1$. Let $\Omega(t_0, \dots, t_k) = \{\text{broken geodesics}\}$
$= \{\omega \mid \omega|_{[t_i, t_{i+1}]} \text{ is a geodesic}\}$.

let $\Omega^a(t_0, \dots, t_k) = \Omega^a \cap \Omega(t_0, \dots, t_k)$

$\mathrm{Int}\,\Omega^a(t_0, \dots, t_k) = \mathrm{Int}\,\Omega^a \cap \Omega(t_0, \dots, t_k)$.

Prop: If M is complete, then $\forall c > 0$ $\exists t_0, \dots, t_k$ s.t. $\mathrm{Int}\,\Omega^c(t_0, \dots, t_k)$ is a finite-dim mfld.

Pf: If $\omega \in \Omega^c$, ~~then~~ then $\ell(\omega|_{[s,t]}) \le \sqrt{(t-s)c}$.
So $\omega([0,1]) \subset B_{\sqrt{c}}(p)$ which is compact. Let $r > 0$ s.t. $B_r(x)$ is a unit normal nbhd $\forall x \in B_{\sqrt{c}}(p)$, choose $t_0, \dots, t_k$ s.t. $|t_i - t_{i-1}| < \frac{r^2}{c}$. So if $\omega \in \mathrm{Int}\,\Omega^c(t_0, \dots, t_k)$, then $\omega([t_{i-1}, t_i])$ is contained in a unit normal nbhd $\Rightarrow$ $\omega([t_{i-1}, t_i])$ determined by endpoints, and ω is determined by $\omega(t_1), \dots, \omega(t_{k-1})$. That is, $\mathrm{Int}\,\Omega^c(t_0, \dots, t_k)$ is homeo to an open subset of M^{k-1}. //

~~If so, then one can show~~ Let $c, t_0, \dots, t_k$ as above, let $B = \Omega^c(t_0, \dots, t_k)$, $B^a = \Omega^a(t_0, \dots, t_k)$ $\forall a < c$. Then:

- E is smooth on B
- $\forall a < c$, B^a is a compact deformation retract of Ω^a
  (namely, given , each seg is in a UNN, so straighten
- Critical pts of E on Ω^a are the critical pts of $E|_B$ on B
  (namely, geodesics)
- If γ is a geodesic, $T_\gamma B = \{\text{broken Jacobi fields}\}$, so
  $\mathrm{index}_E(\gamma) = \mathrm{index}_{E|_B}(\gamma)$, $\mathrm{nullity}_E(\gamma) = \mathrm{nullity}_{E|_B}(\gamma)$.

So: Fundamental Theorem of Morse Theory: If M is complete, and p, q are two points that are not conjugate along any geod of length $\le \sqrt{a}$, then Ω^a is h.e. to a CW complex with one cell of dim λ for every ~~geod from p to q~~ geodesic in Ω^a with index λ.

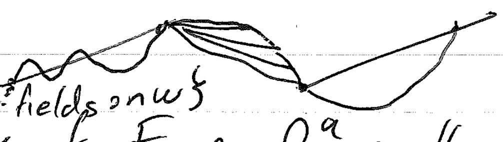
Morse theory on the path space:

Let $\Omega^a = E^{-1}([0,a])$, $\text{Int}\,\Omega^a = E^{-1}([0,a))$. For $0=t_0<t_1<\cdots<t_k=1$,
let $\Omega^a(t_0,\dots,t_k) = \{\text{broken geodesics}\}\cap\Omega^a$
$\text{Int}\,\Omega^a(t_0,\dots,t_k) = \{\text{broken geodesics}\}\cap\text{Int}\,\Omega^a$.

Prop: If M is complete, then $\forall c\ \exists\ 0=t_0<\cdots<t_k=1$ s.t. $\text{Int}\,\Omega^c(t_0,\dots,t_k)$ is a finite-dim. manifold.

Pf: If ~~w∈Ω~~ $w:[0,1]\to M$ is a path, $E(w)<c$, then $l(w|_{[s,t]})<\sqrt{c(t-s)}$.
In particular, $w([0,1])\subset B_{\sqrt{c}}(p)$, which is compact, so $\exists r>0$ s.t. $\forall x\in B_{\sqrt{c}}(p)$, the ball $B_r(x)$ is a UNN. Choose $t_0,\dots,t_k$ so $|t_i-t_{i-1}|<\frac{r^2}{c}$. - then $d(w(t_{i-1}),w(t_i))<r$.
Then if $w\in\Omega^c(t_0,\dots,t_k)$, each seg $w([t_{i-1},t_i])$ lies in a UNN, so $w([t_{i-1},t_i])$ is the unique minimizing geo from $w(t_{i-1})$ to $w(t_i)$.
$\Rightarrow$ w depends smoothly on $w(t_1),\dots,w(t_{k-1})$. ~~Thus, Int $\Omega^c(t_0,\dots,t_k)$ is diffeo to~~ Thus, $w\mapsto(w(t_1),\dots,w(t_{k-1}))\in M^{k-1}$ is a homeo from $\text{Int}\,\Omega^c(t_0,\dots,t_k)$ to an open subset of M^{k-1} //

For c, $t_0,\dots,t_k$ as above, let $B=\Omega^c(t_0,\dots,t_k)$, $B^a=\Omega^a(t_0,\dots,t_k)$ for $0<a<c$. Then: $\forall a$, E is smooth on B^a and B^a is compact
- B^a is a deformation retraction of Ω^a:



(straighten each seg)

- $T_wB=\{\text{broken Jacobi fields on } w\}$
  - Critical pts of E on Ω^a are the crit pts of E on B^a (from proof of Morse Index Thm)
  - $\text{Index}_E(\gamma)=\text{Ind}_{E|_B}(\gamma)$ (from proof of Index Thm) and $\text{nullity}_E(\gamma)=\text{nullity}_{E|_B}(\gamma)$. So:

Fundamental Theorem of Morse Theory: If M is complete, p and $q\in M$ are not conjugate by any geod of ~~energy~~ length $\le\sqrt{a}$, then Ω^a is h.e. to a CW complex with a cell of dim λ for every geodesic in Ω^a with index λ.
~~Follows~~ Pf: $\Omega^a\simeq B^a\simeq$ CW complex.

With some work, can take $a=\infty$ — if p and q not conjugate, then Ω is h.e. to a complex with one λ-cell for each geod from p to q.

Thm (Cartan-Hadamard): Let M be complete, simply-connected, $K(X,Y) \leq 0 \ \forall X, Y \in TM$, then $M \cong \mathbb{R}^n$.

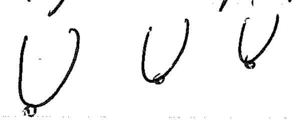
Pf: We saw before that M has no conjugate points, so for $p, q \in M$, every geodesic from p to q has index 0. $\Rightarrow$
$\Omega_{p,q} \simeq$ union of 0-cells.
components of $\Omega_{p,q}$ = # of geodesics from p to q.

But M is simply connected $\Rightarrow$ any two paths from p to q are htpic $\Rightarrow \Omega_{p,q}$ is connected. So there is a unique geodesic from p to q.

So $\exp_p$ is injective, surjective (by Hopf-Rinow), nonsingular, so $\exp_p$ is a diffeomorphism //

Cor: Geodesics from p to q are in 1-to-1 corresp with htpy classes of paths from p to q.

Further, geodesics in M are particularly easy to find — each cpt of $\Omega_{p,q}$ has a single crit pt of E, so the gradient flow on ~~B~~ B^a ~~terminates~~ ends at the ~~sta~~ minimum



~~Thm (Lyusternik~~
~~One more ex~~ Another example: Def: A closed geodesic is a smooth map $\gamma: S^1 \to M$ s.t. $D_t \gamma' = 0$, e.g. equator of sphere. Say ~~we~~ make a bumpy sphere — is there a nontrivial closed geodesic?

Thm (Lyusternik-Fet): If M is a closed Riemannian manifold, then it contains a nontrivial closed geodesic.

Pf: If $\pi_1(M) \neq 0$, then there is a nontrivial free homotopy class of maps $S^1 \to M$ — by the problem set, an energy minimizer in this class is a closed geodesic.

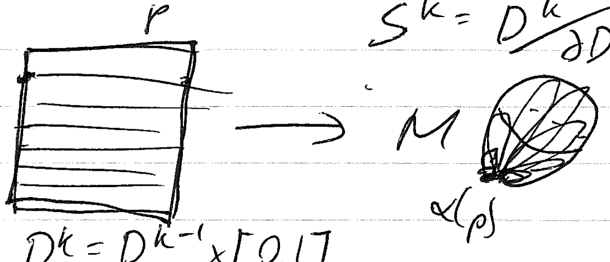
Thus suppose $\pi_1(M) = 0 \Leftrightarrow$ every loop $\gamma: S^1 \to M$ is homotopic to a point. Let ΛM = free loop space of $M = \{f: S^1 \to M \mid f$ piecewise smooth$\}$

Let $E : \Lambda M \to \mathbb{R}$ be energy. E has ∞'ly many trivial crit pts: $E^{-1}(0) = \{\text{constant maps}\} \cong M \subset \Lambda M$.

Any nontrivial crit pt is a closed geodesic (PS) ~~Then ΛM def. retracts to M~~
Suppose there are no nontrivial critical pts. ~~Then ΛM deformation retracts to M. (mean curvature flow of any curve ends at a point)~~
so ~~$\Lambda M \simeq M$ — but some topology~~ shows that $\Lambda M \not\simeq M$ //

Then ΛM def. retracts to M (~~MCF goes to a pt~~ takes ~~any curve to a pt~~ and some others,
Specifically: ~~By the Hurewicz theorem, if $\pi_1(M) = 0$, then~~ because
~~If X is h.e. to a CW complex~~ Then mean curvature flow of any curve ends at a point. But $\Lambda M \simeq M$ $\not\models$ //. To be more clear, need some topology: By some topology, there is a ~~map~~ k s.t.
$\pi_k(M) = \{\text{homotopy classes of maps } S^k \to M\} \neq 0$.
(otherwise, M is contractible — h.e. to a point, ~~which it isn't~~ but compact ~~closed~~ manifolds aren't contractible). Chose ~~k~~ the smallest k s.t. $\pi_k(M) \neq 0$, let $\alpha : S^k \to M$ be htpically nontriv.

$S^k = D^k / \partial D^k \sim p$ ~~Claim: flows~~

α :  M $k = 2$

$\alpha(p)$

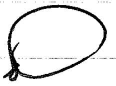
Write $D^k = D^{k-1} \times [0,1]$
Let $\bar{\alpha} : D^{k-1} \longrightarrow \Lambda M$
$q \longmapsto \alpha(q \times [0,1])$

Claim: one of these curves flows to a geodesic.

Sends ∂D^{k-1} to $\alpha(p)$ so $\bar{\alpha} : S^{k-1} \to \Lambda M$. Further ~~$\bar\alpha \sim *$~~ (because if it were ~~Now flow: if no crit pts, then all these curves collapse to points. Then that would give a htpy $\alpha \sim *$~~)

Now flow: no crit pts, so all curves collapse,

$\bar{\alpha}$ is htpic to a map
$\bar{\beta} : S^{k-1} \to E^{-1}(0) = M \subset \Lambda M$.
but $\bar{\beta} \sim *$ by minimality, so
$\bar{\alpha} \sim *$. But a htpy of $\bar{\alpha}$ to $*$



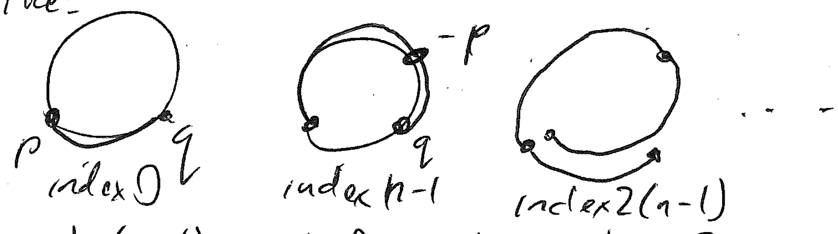
gives a htpy $\alpha \sim *$ $\Rightarrow \times$.

Today: Geometry and topology of spheres.

Spheres: Let $M=(S^n, \text{round})$, $p\in M$. Then, ~~Jacobi p is conjugate to $-p$, p itself~~: geodesics are great circles, p is conjugate to $-p$ and p itself.
(in fact, every unit speed Jacobi field which satisfies $J(0)=0$ also has $J(\pi)=0$, $J(2\pi)=0$, ..., so $\mathrm{ord}_p(-p)=n-1$, $\mathrm{ord}_p(p)=n-1$.)

Fund Thm of Morse theory says that if p,q not conjugate, then $\Omega_{p,q}\simeq X$, where X has a λ-cell for each geod from p to q with index λ.
– So what does ΩS^n look like?

Geods from p to q:



so $\Omega_{p,q}\simeq$ complex with one $k(n-1)$-cell for ~~each~~ all $k\geq 0$.

~~Thm (Morse): Let g be any metric on S^n ($M=(S^n,g)$, where g is ... with $n\geq 3$). Let $p,q\in M$ be non-conjugate. Then $\exists$ inf'ly many geodesics from p to q.~~

Pf: ~~Since $\Omega_{p,q}\simeq$~~ In particular, when $n\geq 3$, it's easy to compute homology of $\Omega_{p,q}$:
$$C_d(X)\xrightarrow{\partial_d} C_{d-1}(X)\xrightarrow{\partial_{d-1}}\cdots\xrightarrow{\partial_2} C_1(X)\xrightarrow{\partial_1} C_0(X)$$
where $C_d(X)$ is ~~generated by the~~ abelian group generated by d-cells of X.
$$C_d(X)=\begin{cases}\mathbb{Z} & \text{if } d=k(n-1)\\ 0 & \text{otherwise}\end{cases}\qquad H_d(X)=\frac{\ker\partial_d}{\mathrm{im}\,\partial_{d+1}}\cong\mathbb{Z} \text{ when } d=k(n-1).$$
So $H_d(\Omega_{p,q})=\mathbb{Z}$ for ∞'ly many d's. And: Let p,q non-conj.

Thm (Morse): Let $M=(S^n,g)$, where g is *any* metric on S^n. Then $\exists$ ∞'ly many geodesics from p to q.

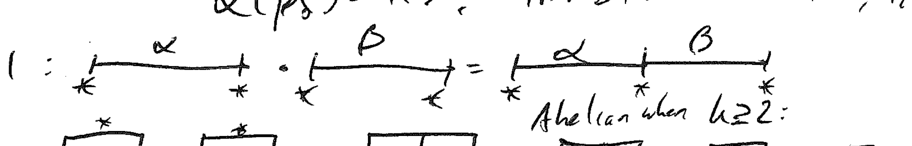
Pf: ~~H_d On one hand~~, By the above, $H_d(\Omega_{p,q})\neq 0$ for ∞'ly many d.
So if $Y\simeq\Omega_{p,q}$, then Y has ∞'ly many cells.
By Fund Thm, $\Omega_{pq}\simeq Y$ where Y has one cell for each geod from p to q.
So ∞'ly many geodesics from p to q. //

Homotopy groups of spheres: Briefly last time: if M a topological space, $*\in S^n$, $*\in M$, $\pi_k(M,*)=\{$homotopy classes of maps $\alpha: S^k\to M$ with $\alpha(*)=*\}$. If M is path connected, $\pi_k(M,*)\cong\pi_k(M,*')$.

These are groups: $k=1$:

Abelian when $k\geq 2$:

$k>1$:



Ex: $\pi_n(S^n) \cong \mathbb{Z}$ ~~Namely: if $\alpha \in S$~~
measures the degree of a map:
if $\alpha: S^n \to S^n$, ~~then~~ and $p \in S^n$ is a regular pt ($\Leftarrow$
~~deg(α) =~~ $D\alpha_x$ nonsingular $\forall x \in \alpha^{-1}(p)$)

then $\deg(\alpha) = \sum_{x \in \alpha^{-1}(p)} \text{sign}(\det D\alpha_x)$ = # of preimages, counted with orientation.

~~is an homomorphism $\pi_n(S^n) \to \mathbb{Z}$.~~
Then $\deg(\alpha)$ is: - independent of p - htpy invar
- if $\deg(\alpha) = \deg(\beta)$, then $\alpha \sim \beta$. - so
$\deg: \pi_n(S^n) \to \mathbb{Z}$ is an isomorphism.

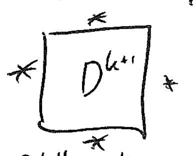
Ex: $\pi_k(S^n) = 0$ ~~$k<n$~~ for all $k < n$.
Pf: WLOG, take $\alpha: S^k \to S^n$ smooth. Then α is not surjective.
Let $p \in S^n$ ~~$\setminus \alpha(S^k)$. Then~~ s.t. $\alpha(S^k) \subset S^n \setminus \{p\}$.
But $S^n \setminus \{p\} \cong \mathbb{R}^n$ - so α is htpic to a point //

~~But~~ Nice and simple. What about $k > n$? <u>Very complicated</u>
$\pi_n(S^2) = \underset{\pi_1}{0}, \underset{\pi_2}{\mathbb{Z}}, \underset{\pi_3}{\mathbb{Z}}, \underset{\pi_4}{\mathbb{Z}_2}, \ldots \mathbb{Z}_2, \mathbb{Z}_{12}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, \ldots$

Ex: $\alpha: S^3 \to S^2$. Let $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$.
$S^2 = \mathbb{C} \cup \{\infty\}$.
Then $\alpha(z,w) = \frac{z}{w}$ is continuous, not null-htpic, called <u>Hopf fibration</u>

Regardless, there are a lot of patterns. One is
Freudenthal Suspension Theorem: If $k < 2n-1$, then $\pi_k(S^n) \cong \pi_{k+1}(S^{n+1})$

Lemma: Let X be a ~~connected~~ space with base point $*$, $k \geq 0$. ~~Then $\pi_{k+1}(X) \cong \pi$~~
Then $\pi_k(\Omega_{*,*} X) \cong \pi_{k+1}(X)$.
Pf: Let $\alpha: S^{k+1} \to X$



$D^{k+1} = D^k \times [0,1]$

$\bar{\alpha}(p) = \alpha|_{p \times [0,1]}$ is a map from S^k to $\Omega_{*,*} X$
and $\alpha \mapsto \bar{\alpha}$ is an isomorphism. //

Pf of Freudenthal: Let $k < 2n-1$.
$$\pi_{k+1}(S^{n+1}) \cong \pi_k(\Omega S^{n+1}) \cong \pi_k(\{p\} \cup D^n \cup D^{2n} \cup \cdots) \cong \pi_k(S^n \vee D^{2n} \cup \cdots)$$

What happens to π_k when we attach ~~D^{2n}~~ a d-cell? ~~a cell of dim $\geq k$~~?

If $d \leq k$, hard to predict.

~~If $2nd \geq k$~~; ~~$\pi_k(X \cup D$~~

If $d = k+1$, ~~$\pi_k(X \cup D^d)$ is a quotient of~~ then ~~there's~~ the map $\pi_k(X) \to \pi_k(X \cup D^d)$ is surjective.

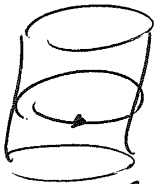
(if $\alpha : S^k \to X \cup D^d$, ~~then $\alpha(S^k)$~~ is smooth on D^d, $\exists\, p \in D^d$ s.t. $\alpha(S^k) \subset X \cup D^d \setminus \{p\} \simeq X$ – so $\alpha \sim \beta$, where $\beta \in \pi_k(X)$).

If $d > k+1$, then $i: \pi_k(X) \to \pi_k(X \cup D^d)$ is an iso: ~~If α is~~ As before, it's surjective. Suppose $\alpha \in \pi_k(X)$, $\alpha \sim *$ by a homotopy $h: S^k \times [0,1] \to X \cup D^d$. Then $\exists\, p \in D^d$ s.t. $\mathrm{im}(h) \subset X \cup D^d \setminus \{p\} \simeq X$. So $\alpha \sim *$ by a homotopy in X $\Rightarrow$ i is injective.

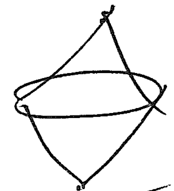
~~So $\pi_k S^n$~~ ~~Since~~ Since $k < 2n-1$,
$\pi_{k+1}(S^{n+1}) \cong \pi_k(S^n \cup D^{2n} \cup D^{3n} \cup \cdots) \cong \pi_k(S^n)$ //

Overflow: In fact, we can construct the map $\pi_k(S^n) \to \pi_{k+1}(S^{n+1})$:

Suspension:



$S^n \times [-1,1]$



$S^n \times [-1,1] / \sim \;=\; \Sigma S^n \cong S^{n+1}$.

If $\alpha : S^k \to S^n$, the <u>suspension</u> $\Sigma\alpha : \Sigma S^k \to \Sigma S^n$, $S^{k+1} \to S^{n+1}$

is the map ~~Σ~~ $\Sigma\alpha(x,t) = (\alpha(x), t)$. ~~preserves~~ sends north pole to north pole, south to south, equator to equator by α.

Freudenthal Suspension Theorem: ~~Σ~~ When $k < 2n-1$, Σ induces an isomorphism $\pi_k(S^n) \to \pi_{k+1}(S^{n+1})$.
When $k = 2n-1$, Σ induces a surjection.

Next time: Comparison geometry.

2024-04-16

Comparison theorems

How does curvature affect geometry of a manifold?
If curvature is larger/smaller than some model, can prove inequalities.
Need: index form:
Def: Let $\gamma : [0,1] \to M$ a geodesic. The index form on γ is (W_1, W_2 piecewise smooth)
$$I(W_1, W_2) = \int_0^1 \langle D_t W_1, D_t W_2 \rangle + \langle R(V, W_1)V, W_2 \rangle \, dt$$

By PS, $I(V_1, V_2) = H(E)(V_1, W_2)$ when $W_i(0) = W_i(1) = 0$
and $I(W_1, W_2) = I(W_2, W_1)$ for all fields (unlike $H(E)$)

geometry is a little tricky — we'll see it's related to growth of Jacobi fields

~~Now: geometry is a little tricky — not as directly connected to energy~~ ~~— same variation field~~
~~Unlike Warning~~: ~~But the relation to energy is trickier, because~~ with
~~A little~~ ~~First~~ Index Lemma: Let γ be a geodesic without conjugate pts, let $W \in \mathcal{V}(\gamma)$ a vector field with $W(0) = 0$. Let J be the Jacobi field with $J(0) = W(0) = 0$, $J(1) = W(1)$. Then $I(J, J) \leq I(W, W)$, with equality iff $J = W$.

~~Idea~~ Pf: ~~Fundamentally~~ By int. by parts, the index form is just 2VF plus a boundary term, an endpoint term. So if we have ~~two~~ fields w/ same endpoint conditions, then we should be able to convert to a 2VF argument:

Let $\bar{J}, \bar{W}$ be extensions of J, W s.t. $\bar{J} = \bar{W}$ on $[1, 1+\varepsilon]$, $\bar{J}(1+\varepsilon) = \bar{W}(1+\varepsilon) = 0$.
~~Then~~ Let $\bar{X} = \bar{W} - \bar{J}$. Then
$$H(\bar{W}, \bar{W}) = H(\bar{J}, \bar{J}) + 2H(\bar{J}, \bar{X}) + H(\bar{X}, \bar{X}).$$

Since $\bar{J}$ is Jacobi on $[0,1]$, $\bar{X} = 0$ on $[1, 1+\varepsilon]$, $H(\bar{J}, \bar{X}) = 0$.
So $H(\bar{W}, \bar{W}) = H(\bar{J}, \bar{J}) + \underbrace{H(\bar{X}, \bar{X})}_{\geq 0 \text{ by index theorem}} \geq H(\bar{J}, \bar{J})$

$\Rightarrow \quad I(\bar{W}, \bar{W}) \geq I(\bar{J}, \bar{J})$
But $I(\bar{W}, \bar{W}) = I(W, W) + I(\bar{W}|_{[1,1+\varepsilon]}, \bar{W}|_{[1,1+\varepsilon]})$
$I(\bar{J}, \bar{J}) = I(J, J) +$ " "
$\Rightarrow I(W, W) \geq I(J, J)$ //

Today, use this to prove:
Rauch Comparison Theorem: Let M, M_0 be Riemannian mflds, with $\dim M_0 \geq \dim M$. Let $\gamma, \gamma_0 : [0, \ell] \to M, M_0$ unit ~~geods~~ speed geodesics with $V = \gamma'$, $V_0 = \gamma_0'$. Suppose ~~$\forall t \in [0,\ell]$~~ γ_0 has no conj. pts and $\forall X \in T_{\gamma(t)}M$, $\forall X_0 \in T_{\gamma_0(t)}M$, $K(X, V) \leq K(X_0, V_0)$.
(Example: M_0 is a plane, M has nonpositive sectional curvature — i.e. gen. curv of M is ~~less~~ $\leq$ curv of M_0.)

If J, J_0 are Jacobi fields on γ, γ_0 with "same initial data": $J_0(0) = J(0) = 0$, $\|D_t J(0)\| = \|D_t J(0)\|$, $\langle V(0) | D_t J(0)\rangle = \langle V_0(0) | D_t J_0(0)\rangle$

($D_t J$ has same length, angle with γ). Then $\|J(t)\| \geq \|J_0(t)\|$ $\forall t \in [0,l]$

(lower curvature, bigger Jacobi fields, stated v. generally) ~~more or less as general~~

Ex: $M_0 = \mathbb{R}^n$, M with $K_M \leq 0$. Let γ a geodesic in M, J a Jacobi, $J(0) = 0$. ~~Then~~ ~~Bec~~ Any Jacobi on M_0 is of form ~~$J_0(t) = J(0) + tD_t$~~

$J_0(t) = J_0(0) + tD_t J_0(0)$, — linear, so ~~$J_0$ with same~~ if J_0 has same initial data, then

$$\|J_0(t)\| = t\|D_t J_0(0)\| = \underline{t\|D_t J(0)\| \leq \|J(t)\|}.$$

J grows superlinearly.

Pf: Write $J' = D_t J$, $J_0' = D_t J_0$. Recall:

$J(t) = J^\perp(t) + \langle V(0) | J'(0)\rangle \cdot t \cdot V(t)$

$\|J(t)\|^2 = \|J^\perp(t)\|^2 + t^2 \langle V(0) | J'(0)\rangle^2$

$\|J_0(t)\|^2 = \|J_0^\perp(t)\|^2 +$ " " — so it suffices to

take $J = J^\perp$, $J_0 = J_0^\perp$.

Let $f(t) = \dfrac{\|J(t)\|^2}{\|J_0(t)\|^2} = \dfrac{\langle J | J\rangle}{\langle J_0 | J_0\rangle}$ Claim that $f(t) \geq 1$ for all t.

ETS: $\lim_{t\to 0} f(t) = 1$ and f is nondecreasing.

$$\lim_{t\to 0} f(t) \overset{L'H}{=} \lim_{t\to 0} \frac{2\langle J | J'\rangle}{2\langle J_0 | J_0'\rangle} = \lim_{t\to 0} \frac{2\langle J' | J'\rangle + \cancel{2\langle J | J''\rangle}_0}{2\langle J_0' | J_0'\rangle + \cancel{2\langle J_0 | J_0''\rangle}_0} = 1.$$

Claim: $f'(t) \geq 0$. ETS $(\log f)'(t) \geq 0$: Need some setup.

$$(\log f)' = \frac{d}{dt} \log\|J(t)\|^2 - \log\|J_0(t)\|^2$$
$$= \underbrace{\frac{\langle J | J'\rangle}{\langle J | J\rangle}} - \underbrace{\frac{\langle J_0 | J_0'\rangle}{\langle J_0 | J_0\rangle}}$$

percentage growth

Normalize: Let $\tau \in [0, l]$, let

$$W(t) = \frac{J(t)}{\|J(\tau)\|}, \quad W_0(t) = \frac{J_0(t)}{\|J_0(\tau)\|}$$

Then $(\log f)'(\tau) = \langle W'(\tau) | W(\tau)\rangle - \langle W_0'(\tau) | W_0(\tau)\rangle$

Claim this is ≥ 0.

Use Index Lemma: First, construct two fields on γ_0:
Let $E_1, \dots, E_n$ parallel orthonormal on γ_1, $E_1 = V$, $E_2(\tau) = W(\tau)$.
$E_1^0, \dots, E_m^0$ " " " on γ_0, $E_1^0 = V_0$, $E_2^0(\tau)' = W^0(\tau)$

Then $W = \sum w^i E_i$ — let $\widehat{W_0} = \sum w^i E_i^0$, so $W_0, \widehat{W_0} \in \mathcal{V}(\gamma_0)$.
Then $\|W\| = \|\widehat{W_0}\| = \sqrt{\sum (w^i)^2}$, $\|W'\| = \|\widehat{W_0}'\| = \sqrt{\sum (w_t^{i})^2}$.

And
$$\langle W_0(\tau) | W_0'(\tau)\rangle = \int_0^\tau \langle W_0' | W_0'\rangle + \langle W_0 | W_0''\rangle \, dt$$
$$= \int_0^\tau \langle W_0' | W_0'\rangle + \langle R(V_0, W_0)V_0 | W_0\rangle \, dt$$
$$= I(W_0, W_0)$$

Compare with:
$$\langle W(\tau) | W'(\tau)\rangle = \int_0^\tau \langle W' | W'\rangle + \langle R(V,W)V | W\rangle \, dt$$
$$= \int_0^\tau \langle W' | W'\rangle - K(V,W) \cdot \|W\|^2 \, dt$$
$$\geq \int_0^\tau \langle \widehat{W_0}' | \widehat{W_0}'\rangle - K(V, \widehat{W_0}) \|\widehat{W_0}\|^2 \, dt$$
$$= I(\widehat{W_0}, \widehat{W_0}) \geq I(W_0, W_0) = \langle W_0(\tau) | W_0'(\tau)\rangle //$$

Uses: Compare fields with different curvature:
~~Lemma: If $K(X,Y) \leq k$ $\forall X, Y \in TM$~~
Lemma: If M, M_0 are such that $K_M \leq K_{M_0}$, i.e. every sect. curv of M is $\leq$ every sect. curv of M_0, $x_0 \in M_0$, $r = \mathrm{injrad}(x_0)$, $x \in M$, and $i : T_{x_0}M_0 \to T_x M$ is an isometry, then the map

$f : B_r(x_0) \to B_r(x)$ is ~~length-increasing~~
$f = \exp_x \circ i \circ \exp_{x_0}^{-1}$ is length-increasing
i.e. $\forall p_0 \in B_r(x_0)$, $\forall v_0 \in T_{p_0} M_0$, $\|Df(v_0)\| \geq \|v_0\|$.

~~Pf: Let $\lambda_0 : (-\varepsilon, \varepsilon) \to M_0$, $\lambda_0'(0) = v_0$, $\tilde{\lambda} = \exp_{x_0}^{-1} \circ \lambda$.~~
~~Let $\alpha_0(u,t) = \exp_{x_0}(t\tilde{\lambda}(u))$ — then $\alpha_0(u,1) = \lambda(u)$~~
~~$\alpha(u,t) = \exp_x(t \cdot i(\tilde{\lambda}(u)))$. $\alpha(u,1) = f(\lambda(u))$.~~

~~Variation fields of α, α_0 are Jacobi fields w/ same initial,~~
~~so $\frac{\partial \alpha}{\partial u}$ $\frac{\partial \alpha}{\partial u}$ so $\left\| \frac{\partial \alpha}{\partial u} \right\|$~~

~~Let $J = \left.\frac{d\alpha}{du}\right|_{u=0}$ $J_0 = \left.\frac{d\alpha_0}{du}\right|_{u=0}(u,t)$ — Jacobi w/ same initial,~~

~~So $\|J(1)\| = \|Df(v_0)\| \geq \|J_0(1)\| = \|v_0\|$. //~~

Leave as exercise

2024-04-18

Last time: Rauch: M, M_0 Riem. mflds, $\dim M_0 \ge \dim M$, γ, γ_0 unit-speed geods, $V=\gamma'$, $V_0=\gamma_0'$. $\forall t, \forall X \in T_{\gamma(t)}M, X_0 \in T_{\gamma_0(t)}M$, $K(V(t),X) \le K(V_0(t),X_0)$

If J, J_0 Jacobi fields on γ, γ_0 with same initial data ($J(0)=J_0(0)=0$, $\|D_tJ(0)\| = \|D_tJ_0(0)\|$, $\langle V | D_tJ(0)\rangle = \langle V_0 | D_tJ_0(0)\rangle$), then $\|J(t)\| \ge \|J_0(t)\|$.

~~Main use~~ Application: exponential maps: if ~~$\forall \in T_xM$~~ $V, W \in T_xM$, ~~then~~ $\gamma(t) = \exp(Vt)$, $J \in \mathcal{V}(\gamma)$ a Jacobi field s.t. $J(0)=0$, $D_tJ(0)=W$, then $J(t) = D\exp_{tV}(tW)$ $\forall t$ — so if $J(t) \ge J_0(t)$, then $\|D\exp_{tV}(W)\| \ge \|D\exp_{tV_0}(W_0)\|$. Thus:

Lemma: Let M, M_0 of same dimension s.t. $K_M \le K_{M_0}$. Let $x \in M$, $x_0 \in M_0$, let $i: T_{x_0}M_0 \to T_xM$ an isom, let $r < \mathrm{inj\,rad}(x_0)$. Then $f: B_r(x_0) \to B_r(x)$, $f = \exp_x \circ i \circ \exp_{x_0}^{-1}$ is length-increasing.

Today: Use this to show: If M is complete and simply connected, $K_M \le 0$, then M is a CAT(0) space. This is sthing we can define for metric spaces.

~~Let X be a metric space~~

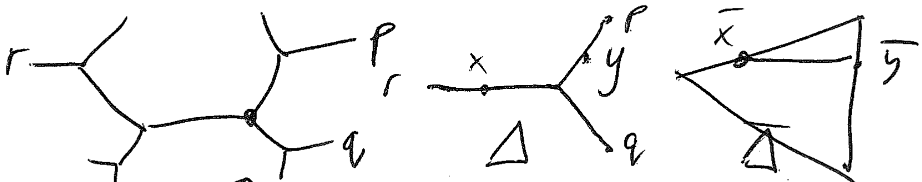
CAT(0) spaces: ~~A geodesic triangle is a loop made of three geodesic segments [$\gamma: [a,b]$~~

Let X be a metric space. A geodesic in X is a map ~~$\gamma:[a,b]$~~ $\gamma: I \to X$ s.t. $d(\gamma(s), \gamma(t)) = |s-t|$ $\forall s,t \in I$.
(notably, unit speed, length-minimizing)

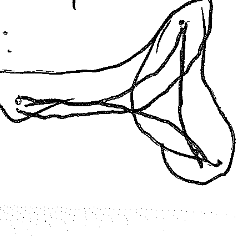
A geodesic triangle is a loop made of three geodesic segments — we write $[p,q]$ for edges, $\Delta(p,q,r)$ for the triangle. If Δ is a geodesic triangle, the comparison triangle $\bar\Delta = \bar\Delta(\bar p, \bar q, \bar r)$ is the triangle in $\mathbb{R}^2$ with same side lengths. If $x \in [p,q]$, the comparison point $\bar x$ is the pt s.t. $d(\bar p, \bar x) = d(p,x)$.

X is CAT(0) if ~~the~~ it is geodesic ($\forall x,y$, $\exists$ geod from x to y), and if $\forall$ geod tri Δ, $\forall x,y \in \Delta$, $d(x,y) \le d(\bar x, \bar y)$ — every chord of Δ is shorter than corresponding chord of $\bar\Delta$.

Prototypes: - X is a tree:



- X is the hyperbolic plane:



triangles in hyp plane are "thin": third edge is contained in nbhd of other two.

In fact, ~~$K_M \leq 0 \Rightarrow M$ is~~

Thm: ~~Suppo~~ Let M a complete Riem mfld, $K(X,Y) \leq 0$ $\forall X, Y \in T_pM$. Then M is CAT(0).

Won't give full proof, will rely on following thm about metric spaces:

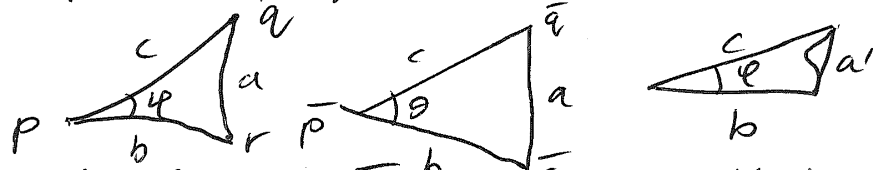
Thm: ~~X is CAT~~ Let X a metric space. TFAE:

- X is CAT(0)
- $\forall \Delta(p,q,r)$, $\forall x \in [q,r]$, $d(p,x) \leq d(\bar{p},\bar{x})$.
- $\forall \Delta(p,q,r)$, if p,q,r are distinct then $\angle_p \leq \angle_{\bar{p}}$,

where $\angle_p$ is angle at p if X is a mfld, and otherwise, $\angle_p(\gamma,\gamma')$ is defined by $\cos\angle_p(\gamma,\gamma') = \liminf\limits_{t,t'\to 0} \dfrac{t^2+(t')^2-d(\gamma(t),\gamma'(t'))^2}{2tt'}$.

Pf: Omitted (see Bridson–Haefliger). //

Given this, Pf of Thm: Let $\Delta = \Delta(p,q,r)$. By C-H, $\exp_p$ is a diffeo from $T_pM \to M$, by Rauch, $\exp_p$ is length-increasing. So: draw 3 "triangles"



$\Delta \subset M$ $\quad \bar{\Delta} \subset \mathbb{R}^2$ $\quad \exp_p^{-1}(\Delta)$

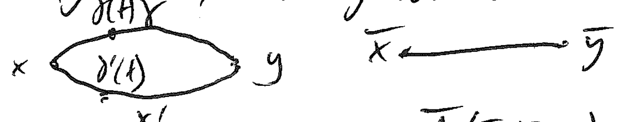
Then $a' \leq \ell(\exp^{-1}([q,r])) \leq \ell([q,r]) = a$

$$\Rightarrow \cos\theta = \frac{c^2+b^2-a^2}{2bc} \leq \frac{c^2+b^2-(a')^2}{2bc} = \cos\varphi$$

$$\Rightarrow \angle_p \leq \angle_{\bar{p}}. \quad \text{Thus } M \text{ is CAT(0). } //$$

Once X is CAT(0), several properties:

- $\forall x,y \in X$, $\exists!$ geodesic from x to y.

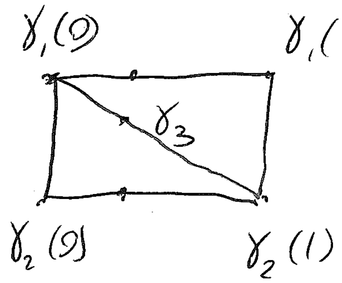


$\Delta(x,y,y)$ $\quad \bar{\Delta}(\bar{x},\bar{y},\bar{y})$ $\quad$ - so $d(\gamma(t),\gamma'(t)) \leq d(\bar{\gamma}(t),\bar{\gamma}'(t)) =$

- X is contractible (identity map is homotopic to constant map): Let $x \in X$ be a basepoint. $\forall y \in X$, let γ_y be a constant-speed geodesic from y to x, $\gamma_y(0)=y$, $\gamma_y(1)=x$. Then $h_t(y) = \gamma_y(t)$ is a homotopy from id_X to const.

- Prop: X has a convex distance function: $\forall \gamma,\gamma' : [0,1] \to X$ constant speed geodesic, $d(\gamma(t),\gamma'(t))$ is convex.

Pf:



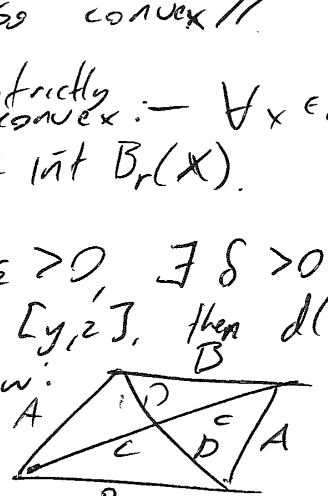
Let γ_3 a geod from $\gamma_1(0)$ to $\gamma_2(1)$.
Then $d(\gamma_1(t), \gamma_2(t)) \leq d(\gamma_1(t), \gamma_3(t)) + d(\gamma_3(t), \gamma_2(t))$.
$\leq t\, d(\gamma_1(1), \gamma_2(1)) + (1-t)\, d(\gamma_1(0), \gamma_2(0))$.
So convex //

- Every metric ball is strictly convex: — $\forall x \in X$, $r>0$, $y, z \in B_r(x)$, $[y,z] \subset B_r(x)$ and $(y,z) \subset \operatorname{int} B_r(X)$.

Follows from:

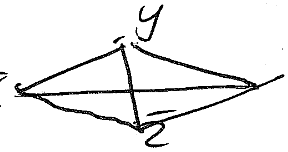
Lemma: $\forall x \in X$, $\forall r, \varepsilon > 0$, $\exists \delta > 0$ s.t. if $y, z \in B_r(x)$, $d(y,z) > \varepsilon$, m is midpoint of $[y,z]$, then $d(x,m) < r - \delta$.

Pf: Parallelogram law:



$A^2 + B^2 = 2(C^2 + D^2)$.

So we can draw comparison triangle:
and extend to comparison parallelogram:



If $A, B \leq r$, $D \geq \varepsilon/2$, then
$$2C^2 + \frac{\varepsilon^2}{2} \leq 2r^2 \Rightarrow C < \sqrt{r^2 - \frac{\varepsilon^2}{4}} = r - \Omega\left(\frac{\varepsilon^2}{r}\right).$$

By CAT(0), $d(\bar{x}, \bar{m}) \leq C$ //.

Overflow: Thm (Cartan): Let X be CAT(0), let $\Gamma \leq \operatorname{Isom}(K)$. If Γ has a bounded orbit, then Γ fixes a point.

Need a def, lemma:

~~Lemma~~ Def: Let $Y \subset X$. The radius of Y is
$$r_Y = \inf \{ r \mid \exists x \in X \text{ s.t. } Y \subset \overline{B}_r(x) \}.$$

Lemma: If $Y \subset X$ is bounded nonempty, then $\exists!\, c_Y \in X$ s.t. $Y \subset \overline{B}_{r_Y}(c_Y)$ — i.e., infimum is achieved for a unique center.

Pf: Let $x_1, x_2, \ldots \in X$, $r_1, r_2, \ldots > 0$ s.t. $Y \subset B_{r_i}(x_i)$ and $\lim r_i = r_Y$.

Claim: x_i is Cauchy. Pf: Let $\varepsilon > 0$, let $N > 0$ s.t. $r_i - r_X < \varepsilon$ $\forall i > N$.

Let $i, j > N$, let $m =$ midpt of $[x_i, x_j]$. Then $\forall y \in Y$
$d(x_i, y) < r_Y + \varepsilon$, $d(x_j, y) < r_Y + \varepsilon$, so $d(m,y) < r_Y + \varepsilon - \Omega\left(\frac{d(x_i,x_j)^2}{r_Y}\right)$
By minimality, $r_Y \leq r_Y + \varepsilon - \Omega\left(\frac{d^2}{r}\right) \Rightarrow d(x_i, x_j) = O(\sqrt{\varepsilon r_Y}) \Rightarrow$ Cauchy,
can take $c_Y = \lim x_i$.
Further, if $Y \subset \overline{B}_{r_Y}(c')$, then $c_Y, c', c_Y, c', \ldots$ is also Cauchy $\Rightarrow c_Y = c'$. //

Ask: deRham, Hodge, on lie groups, symmetric spaces?