2025-03-17

Last time: Applications of Jacobi fields.

~~Thm (Bonnet-Myers)~~ State next one, we need:

Def: $\forall p \in M$, if $E_1, \dots, E_n \in T_pM$ is an o.n basis, $U_1, U_2 \in T_pM$, let $Ric(U_1, U_2) = \sum_i \langle R(E_i, U_1)U_2 | E_i \rangle$
$= tr(E \mapsto R(E, U_1)U_2)$

This is the Ricci curvature of M – a symmetric bilinear form.
In particular, if X is a unit vector, we can take $E_1 = X$, then

$$Ric(X,X) = \sum_i \langle R(E_i, X)X | E_i \rangle = \sum_{k=2}^{n} K(X, E_k)$$

= average sectional curvature.

Last time I stated this in terms of sectional curv, but really about Ricci: Thm (Bonnet-Myers): Suppose M is an n-manifold, $r>0$, and $\forall U \in TM$, $\|U\|=1$, then $Ric(U,U) \geq \frac{n-1}{r^2}$ (For instance, if $K(X,Y) \geq \frac{1}{r^2}$ $\forall X, Y$). Then any geodesic of length $> \pi r$ is not minimizing.

Pf: Let $\gamma: [0,1] \to M$ a geodesic of length $L \geq \pi r$.
Let ~~$E_0 = \frac{\gamma'}{\|\gamma'\|}$, let~~ $E_1, \dots, E_n$ orthonormal fields s.t. $E_1 = \frac{\gamma'}{\|\gamma'\|}$.
For $i = 2, \dots, n$, let $W_i = \sin(\pi t) E_i$. Claim: $H(W_i, W_i) \leq 0$ for some i.

$$H(W_i, W_i) = -\int_0^1 \langle \underbrace{W_i}_{\sin(\pi t)E_i} | \underbrace{D_t^2 W_i}_{-\pi^2 \sin(\pi t) E_i} - R(\gamma', W_i)\gamma' \rangle \, dt$$

$$= \int_0^1 \pi^2 \sin^2(\pi t)\,dt + \int_0^1 \langle W_i | R(\gamma', W_i)\gamma' \rangle \, dt$$
$$= \frac{\pi^2}{2} - \int_0^1 \|W_i\|^2 \|\gamma'\|^2 K(E_1, E_i)\,dt$$

So
$$\sum_{i=2}^{n} H(W_i, W_i) = (n-1)\frac{\pi^2}{2} - \int_0^1 \sin^2(\pi t) L^2 Ric(E_1, E_1)\,dt$$
$$\leq (n-1)\frac{\pi^2}{2} - \frac{1}{2} L^2 \frac{(n-1)}{r^2}$$. If $L > \pi r$, then this is < 0 $\Rightarrow$ γ is non-minimizing //

Cor: If M is complete, then $\mathrm{diam}(M) \leq \pi r$.

~~Polar coordinates~~: ~~Bekic~~ ~~Son~~

Other things today: Normal coordinates:

Def: Let $U \subset T_pM$ s.t. $\exp_p$ is a diffeo on U. We call $(\exp_p|_U)^{-1}: M \to U$ a normal coordinate chart centered at p.

Then we can ~~often calculate~~ + use Jacobi fields, curv: to calculate metric in normal coords.

Ex: Suppose $M = \cancel{M_K^n}$ ~~model space of~~ has dim 2, has constant sectional curvature K. We expect the metric to be rotationally symmetric, so we consider polar coords $(r, \theta) \in T_pM$, $r \geq 0$, $\theta \in [0, 2\pi]$.
Then ∂_r is radial vector, since geods have constant speed, $\|\partial_r\| = 1$. By Gauss's Lemma, $\langle \partial_r, \partial_\theta \rangle = 0$. ~~Consider~~ So ~~it suffices to~~ it remains to calculate $\|\partial_\theta\|^2$.
Let N be o.n. to ∂_r. Then
But: ~~For any θ,~~ $J(r) = \partial_\theta$ is ~~a normal~~ the Jacobi field: with $\|D_t J(0)\| = N$, $J(0) = 0$. ~~with $D_t J(0)$~~ ~~with $\|D_t J(0)\|$~~ (corresponds to a variation through geodesics). So

$$D_t^2 [\partial_\theta] = R(\partial_r, \partial_\theta)\partial_r$$

leave as ex.

Let N be o.n. to ∂_r - then N is parallel, $\partial_\theta = f(r) N$.

$$D_t^2 \partial_\theta = f''(r) N. \qquad R(\partial_r, \partial_\theta)\partial_r = f(r) R(\partial_r, N)\partial_r$$
$$= -f(r) K(\partial_r, N) \cdot N$$
$$= -K f(r) N.$$

~~So $f''(r) = -Kf(r)$~~ Exercise: Show that
~~So if N is o.n. to ∂_r. Then~~ $J(r) = f(r) N$ where $f''(r) = -K f(r)$, $f(0) = 1$

$$\text{i.e. } f(r) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K} r) & K > 0 \\ r & K = 0 \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{K} r) & K < 0. \end{cases} \quad \text{and } ds^2 = dr^2 + f(r)^2 d\theta^2.$$

~~So This proves sth we mentioned a while ago: there's a unique complete, simply connected model space with const sectional curv K.~~
~~Pf: When $K \leq 0$, $M = \mathbb{R}^2$ with metric $ds^2 = dr^2 + f(r)^2 d\theta^2$.~~
~~When $K > 0$~~ simply connected, geod complete

Recall: $\exists!$ model space with const. sect. curv K.
Pf: Let P be such a space. ~~Equip~~ let $p \in P$, equip T_pM with metric g above, consider $\exp_p : (T_pM, g) \to P$. If $K \leq 0$, then $\exp_p$ is a local ~~isometry~~ diffeo: $\forall v \in T_pM$, $\exists$ a nbhd $U \ni v$ s.t. $\exp_p|_U$ is a ~~metric-preserving~~ diffeo. ~~Since P is s.c., this is a global~~ diffeo on all of In particular, $\exp_p$ is a covering map, so since P is s.c. it is a diffeo.
Then exp preserves metric.

When $K > 0$, ~~one more step~~ slightly diff. Let
Pf: Let P be such a space. Let M_K^n be the ~~models~~ space, let $p \in P$, $m \in M_K^n$ ~~$\alpha : M_K^n \to P$ be the map exp~~, let $I : T_mM \to T_pP$ an isometry. $K \leq 0$ let $\alpha : M \to P$

$$\alpha(x) = \exp_p(I(\exp_m^{-1}(x)))$$

This is metric-preserving. If $K \leq 0$, this is a ~~local diffeo~~

covering map, but P is simply connected, so its a diffeo.
If $\kappa > 0$, need to be more careful: ~~det exp~~
let $\alpha(x) = \exp_p \circ I \circ \exp_m|_{B_{1/\sqrt{\kappa}}}(x)$.

Careful with this, it's a little unclear

This is well-defined everywhere except $-x$, and metric preserving, so we can extend to x by continuity. Then α is metric-preserving at $-x$ too $\Rightarrow$ covering map $\Rightarrow$ isometry. //

<u>Normal coordinates</u>: let $p \in M$, let $E_1, \dots, E_n$ an o.n basis of T_pM.
Let $r < \mathrm{injrad}(p)$, let $\alpha : B_r(0) \to M$
$$\alpha(x^1, \dots, x^n) = \exp_p(x^i E_i).$$
Then α is a diffeo and $\alpha^{-1} : \exp(B_r(0)) \to \mathbb{R}^n$ is a <u>normal coord chart</u>

Thm: 1) Lines through 0 go to geodesics.
2) By Gauss, spheres around 0 map to metric spheres
3) For any geod $\gamma_V(t) = \exp_p(tV)$, the fields ~~$t\partial_i$~~ $J_i(t) = t\partial_i$ are Jacobi fields.
4) So we can estimate $g_{ij} = \langle \partial_i | \partial_j \rangle$ based on curvature!
~~$g_{ij} = \langle \partial_i | \partial_j \rangle = t^{-2} \langle J_i(t) | J_j(t) \rangle$~~
$$\langle J_i(t) | J_j(t) \rangle = t^2 \langle \partial_i | \partial_j \rangle = t^2 g_{ij}(\cancel{\exp}\, tV).$$
Expand w/ Taylor series: Let $f_{ij} = \langle J_i | J_j \rangle$. Then
$$f_{ij}^{(k)} = \sum_m \langle D_t^m J_i | D_t^{k-m} J_j \rangle$$
We can calculate:

$$\begin{aligned}
J_i &= t\partial_i & J_i(0) &= 0 \\
D_t J_i &= \partial_i + t D_t \partial_i & D_t J_i(0) &= \partial_i \\
D_t^2 J_i &= R(V, J_i)V = tR(V, \partial_i)V & D_t^2 J_i(0) &= 0 \\
D_t^3 J_i &= R(V, \partial_i)V + t \tfrac{d}{dt}[R(V, \partial_i)V] & D_t^3 J_i(0) &= R(V, \partial_i)V.
\end{aligned}$$

Then $f_{ij}(0) = 0$, $f_{ij}'(0) = 0$. $f_{ij}''(0) = 2\langle \partial_i | \partial_j \rangle = 2\delta_{ij}$.

$$f_{ij}'''(0) = 0, \quad f_{ij}^{(4)}(0) = 4\langle D_t J_i(0) | D_t^3 J_j(0) \rangle + 4\langle D_t^3 J_i(0) | D_t J_j(0) \rangle = 8\langle R(V, \partial_i)V | \partial_j \rangle.$$

So
$$t^2 g_{ij}(tV) = \delta_{ij} t^2 + \frac{8}{24} \langle R(V, \partial_i)V | \partial_j \rangle t^4 + O(t^5)$$
$$g_{ij}(tV) = \delta_{ij} + \frac{1}{3} \langle R(V, \partial_i)V | \partial_j \rangle t^2 + O(t^3)$$
This is close to Ricci ~~– in fact, say $V = \partial_1$~~. consider $B(V, \varepsilon) \subset T_pM$
Then $\mathrm{vol}(\exp_p(B(V, \varepsilon)))^2 \approx |\det(g_{ij})|^2 \mathrm{vol}\, B_{\varepsilon}^2$
~~$(g_{ij}) = I + \frac{1}{3}t^2(\mathrm{Ric}(\partial_i, \partial_j))$~~

~~$\approx 1 - \frac{t^2}{3}\mathrm{Ric}(\partial_1, \partial_1)$~~
$$\approx 1 - \frac{t^2}{3} \mathrm{Ric}(V, V)$$
$$\Rightarrow \mathrm{vol}(\exp_p(B(V, \varepsilon))) / \mathrm{vol}\, B(V, \varepsilon) \approx 1 - \frac{t^2}{6} \mathrm{Ric}(V, V)$$

2025-03-31

Today: Morse Index Theorem.

~~Generalizes two results~~

Previously: Thm: If γ is a length-minimizing geodesic, then its interior contains no conjugate points.

Morse Index Thm is a converse and generalization. Need some defs:

~~Prop~~: Let F be a symmetric bilinear form on V. Then $\exists$ a decomp $V = V_- \oplus V_0 \oplus V_+$ s.t.

- F is <u>positive definite</u> on V_+: $F(v,v) > 0 \ \forall v \in V_+ \setminus 0$.
- F is <u>negative definite</u> on V_-: $F(v,v) < 0 \ \forall v \in V_- \setminus 0$
- $V_0 = \mathrm{null}(F) = \{v_0 \mid F(v,v_0) = 0 \text{ for all } v \in V\}$.

{Further, if $W_-, W_+ \subset V$, $F|_{W_-} < 0$, and $F|_{W_+} > 0$, we can choose $V_- \supset W_-$, $V_+ \supset W_+$ as above.

- if $V_+ \oplus V_- \oplus V_0$ and $V_+' \oplus V_-' \oplus V_0'$ are two such decomps, then $\dim V_\pm = \dim V_\pm'$.

Pf: Exercise //

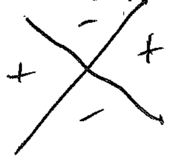
Def: $\mathrm{index}(F) = \dim V_- =$ max dim of a neg-det subspace.

$\mathrm{nullity}(F) = \dim V_0$

In particular, if $\mathrm{index}(F) = 0$, then $Q(v) = F(v,v)$ has local min at 0.

$\boxed{\text{Take this slowly}}$

Ex: $F(v,v) = x^2 - y^2 \ \forall v = (x,y) \in \mathbb{R}^2$.



V_+? V_-? Note $-$ $F(v,v) = 0$ isn't the same as $v \in V_0$.

Recall: If fin. $V = \mathbb{R}^n$, then $\exists$ o.n. basis s.t. $F(x,x) = \sum_i \lambda_i (x^i)^2$

~~null index(F) = # of negative eigvals~~

Recall: If γ is a geodesic, $\gamma(0) = p$, $\gamma(t) = q$, then

$$\mathrm{ord}_p(q) = \dim\{J \mid J \text{ is Jacobi}, J(0) = 0, J(t) = 0\} = \dim(\mathrm{null}(H(\gamma))).$$

Thm (Morse Index Theorem):

Let $\gamma : [0,1] \to X$ be a geodesic. Let $\mathrm{index}(\gamma) = \mathrm{index}_{H(E)}(\gamma)$.

Then ~~index(γ) there are finitely many points on γ with nonzero index, and index(γ) = $\sum_{t\in[0,1)} \mathrm{ord}_{\gamma(0)}(\gamma(t))$~~. $\mathrm{index}(\gamma) < \infty$ and

$$\mathrm{index}(\gamma) = \sum_{t \in [0,1)} \mathrm{ord}_{\gamma(0)}(\gamma(t)).$$

(This might take two lectures, so ~~I want to~~ to go over strategy:

1 - Show that $\mathrm{index}(\gamma)$ is finite: find a large space $J^\perp$ s.t. $F|_{J^\perp} > 0$.

2 - Show that $\mathrm{index}(\gamma)$ is increasing and only increases at conj pts.

3 - Show that $\mathrm{index}(\gamma)$ increases at each conj pt.

~~Pf~~ Step 1: Reduce to a finite-dim subspace.

~~Lemma~~: Lemma: $\exists$ a finite-dim subsp $J \subset T_\gamma\Omega$ and a $J^\perp \subset T_\gamma \Omega$ s.t.

① $T_\gamma\Omega = J + J^\perp$ ② $J \cap J^\perp = 0$ ③ $H(J, J^\perp) = 0$

④ $H|_{J^\perp} > 0$ ⑤ ~~Ea~~ $\text{index}(H|_J) = \text{index}(H)$ ⑥ $\text{null}(H) = $ ~~Jacobi~~ ~~Jacobi~~ $\{W \mid W \text{ Jacobi } W(0)=0, W(1)=0\}$

Pf: Let $\varepsilon > 0$ small enough that $\forall t \in [0,1]$, ~~$B_\varepsilon(\gamma(t))$~~ $\subset J$.
$B_{2\varepsilon}(\gamma(t))$ is contained in a unif normal nbhd. Let
$0 = t_0 < t_1 < \cdots < t_k = 1$ s.t. $|t_i - t_{i-1}| < \varepsilon$.
Let $J = J(t_0, \ldots, t_k) = \{W \in T_\gamma\Omega \mid W|_{[t_i, t_{i+1}]} \text{ is Jacobi for all } i\}$
$= \{$broken Jacobi fields$\}$.
Let $J^\perp = \{W \in T_\gamma \Omega \mid W(t_i) = 0 \; \forall i\}$.
~~Then $W \in J$ is determined by $W(t_0), W(t_1), \ldots, W(t_k) \Rightarrow \dim(J) < \infty$~~
Then: Each segment ~~has been~~ is contained in a UNN, so
$\gamma|_{[t_0, t_1]}$ is minimizing and $\gamma(t_i)$ is not conjugate to $\gamma(t_{i+1})$.
Therefore, any $W \in J$ is determined by $W(t_0), \ldots, W(t_k) \Rightarrow \dim(J) < \infty$.
①, ②: Clear. ③: 2VF: if $W \in J$, $X \in J^\perp$,
$$H(W,X) = -\sum_t \langle \underbrace{X}_{0} \mid \Delta_t D_t W\rangle - \int_0^1 \langle X \mid \underbrace{D_t^2 W - R(\gamma', W)\gamma'}_{0} \rangle \, dt = 0.$$

④ $H|_{J^\perp} > 0$. Let $X \in J^\perp$, let $X_i = X|_{[t_i, t_{i+1}]}$. Then $\gamma|_{[t_i, t_{i+1}]}$ is minimal, so $H(X_i, X_i) \geq 0$.
Suppose $H(X_i, X_i) = 0$. ~~On one hand~~ Claim $X_i = 0$. On one hand,
$H(X_i, J) = 0$. OTOH, ~~suppose~~ for all $Y \in J^\perp$,

$$g(\varepsilon) = H(X + \varepsilon Y, X + \varepsilon Y) = H(X,X) + 2\varepsilon H(X,Y) + \varepsilon^2 H(Y,Y)$$
$$= 2\varepsilon H(X,Y) + \varepsilon^2 H(Y,Y) \geq 0 \;\forall \varepsilon.$$
$\Rightarrow H(X,Y) = 0$. So $H(X, J^\perp) = 0 \Rightarrow X \in \text{null}(H) = J$
$\Rightarrow X \in J \cap J^\perp = 0$ //

~~So Clearly~~ This lets us bound $\text{index}(\gamma)$. Let $p, p^\perp$ projections from $T_\gamma\Omega$ to $J, J^\perp$.
Then $\forall W \in T_\gamma \Omega$, $H(W,W) = H(p(W), p(W)) + \underbrace{2H(p(W), p^\perp(W))}_{0} + \underbrace{H(p^\perp(W), p^\perp(W))}_{\geq 0}$
$$\geq H(p(W), p(W)).$$
So if $H|_N < 0$, then $H|_{p(N)} < 0$ ~~and~~ $N \cap J^\perp = 0$, so $\dim(p(N)) = \dim N$ and $H|_{p(N)} < 0$. $\Rightarrow$
$\text{index}(H) = \text{index}(H|_J) \leq \dim(J) < \infty$.
Likewise, $\text{null}(H)$ ~~= Jacobi~~ $\subset J$.

So it suffices to consider $H|_J$.

Step 2: Let $\gamma_\tau = \gamma|_{[0,\tau]}$, $H_\tau = H(E)_{\gamma_\tau}$, $\lambda(\tau) = \text{index}(H_\tau)$

~~Lemma:~~ ~~then~~ How does $\lambda(\tau)$ depend on τ?

① $\lambda(\tau)$ is nondecreasing. (a v. field on γ_τ extends to a field on $\gamma_{\tau+\varepsilon}$.)

② $\lambda(\tau) = 0$ for $\tau < \varepsilon$: index H_τ = index $H|_{J(0,\varepsilon)}$, but $J(0,\varepsilon) = 0$.

Claim: λ only changes at conjugate pts.

Lemma: $\forall \tau$, $\lambda(\tau - \delta) = \lambda(\tau)$ for small δ.

Pf: Partition so $0 = t_0 < \cdots < t_i \leq \tau < t_{i+1}, \cdots$.

Let $J_\tau = J(t_0, \ldots, t_i, \tau)$ ~~Doesn't ... depe~~

Recall $J_{\tau-\delta} \cong (T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_i)}M)$ when $\tau - \delta > t_i$.

Call this Σ.

Then $H_{\tau-\delta}$ is a form on Σ that varies ctsly.

If $N \subset \Sigma$ and $H_\tau|_N < 0$, then $H_{\tau-\delta}|_N < 0$ for small δ.

$\Rightarrow \lambda(\tau - \delta) \geq \lambda(\tau)$, but λ is increasing, so $\lambda(\tau-\delta) = \lambda(\tau)$.

So where can λ change? Recall: $\Sigma = \Sigma_+ \oplus \Sigma_0 \oplus \Sigma_-$.

If we perturb H to $\bar{H}$, then $\bar{H}|_{\Sigma_+} > 0$, $\bar{H}|_{\Sigma_-} < 0$

- only Σ_0 can change sign.

~~Lemma:~~

So: Lemma: Let $\tau \in [0,1]$, $n = \text{nullity}(H_\tau)$. Then $\forall$ suff small $\delta > 0$,

$\lambda(\tau + \delta) \leq \lambda(\tau) + n$.

Pf: Let Σ as above, let $\Sigma = \Sigma_0 \oplus \Sigma_+ \oplus \Sigma_-$ (wrt H_τ). If δ suff small, then $H_{\tau+\delta}|_{\Sigma_+} > 0$. So we can decomp $\Sigma = \Sigma_0' \oplus \Sigma_+ \oplus \Sigma_-'$ (wrt $H_{\tau+\delta}$).

$\Rightarrow$ ~~$\dim(\Sigma_0')$ nullity$(H_{\tau+\delta})$ + index$(H_{\tau+\delta}) \leq \dim + \dim$~~

$\dim(\Sigma_-') \leq \dim(\Sigma_0) + \dim(\Sigma_-) = \lambda(\tau) + n$. //

2025-04-02

Last time: Morse Index Thm: Let $\gamma:[0,1]\to M$ a geodesic. Then $\mathrm{index}(H_\gamma(E)) = \sum_{t\in(0,1)} \mathrm{ord}_{\gamma(0)}\gamma(t)$, where
$\mathrm{index}(H) = \sup\{\dim W \mid W\subset T_\gamma\Omega,\ H(V,W)<0\ \forall W\in N\setminus 0\}$.
$(H|_N<0)$.

and $\mathrm{ord}_{\gamma(0)}\gamma(t) = \dim\{W\in T_\gamma\Omega \mid V \text{ is Jacobi}\}$.

So far: If $0=t_0<t_1<\cdots<t_k=1$ and $|t_{i+1}-t_i|$ suff small, and $J(t_0,\dots,t_k)=\{W\in T_\gamma\Omega \mid W \text{ is Jacobi on } [t_i,t_{i+1}]\}$, then $\mathrm{index}\, H = \mathrm{index}\, H|_J$.

Next: Let $\gamma_\tau = \gamma|_{[0,\tau]}$, $H_\tau = H_{\gamma_\tau}$, $\lambda(\tau)=\mathrm{index}\, H_\tau$.
How does λ depend on τ?

① λ is nondecreasing: if $V\in T_{\gamma_\tau}\Omega$, $H_\tau(W,V)<0$, then $\forall h>0$, can extend W by 0 to get $\overline{W}\in T_{\gamma_{\tau+h}}\Omega$ with $H_{\tau+h}(\overline{W},\overline{W})<0$.
So if $H_\tau|_N<0$, then $H_{\tau+h}|_{\overline{N}}<0$.

② Let $n=\mathrm{nullity}(H_\tau)=\mathrm{ord}_{\gamma(0)}\gamma(\tau)$. If h is suff small, then $\gamma(\tau+h)\in[\gamma(\tau),\gamma(\tau)+n]$.

Pf: Suppose $t_i<\tau<t_{i+1}$ (otherwise, adjust t_i's).
Let $J_{\tau+h} = J(t_0,\dots,t_i,\tau+h)$ for small h.
We chose t_i's so $J_{\tau+h}\cong T_{\gamma(t_1)}M\oplus\cdots\oplus T_{\gamma(t_{k-1})}M = V$
View $H_{\tau+h}$ as a family of forms on V.
Then $V=V_+\oplus V_0\oplus V_-$ where $H_\tau|_{V_+}>0$, $V_0=\mathrm{null}(H_\tau)$, $\dim(V_-)=\lambda(\tau)$.

If h small, then $H_{\tau+h}|_{V_+}>0$ by cty. So
$\mathrm{index}(H_{\tau+h})\le \dim V-\dim V_+ = \dim V_0+\dim V_- = \lambda(\tau)+n$.

That is, λ only increases at conj pts.

Step 3: λ increases at every conj pt: if $h>0$ suff small, then $\lambda(\tau+h)=\lambda(\tau)+n$.
Pf: Let $k=\mathrm{index}(H_\tau)$, let $W_1,\dots,W_k\in T_{\gamma_\tau}\Omega$ generate a neg-def subsp. Let $J_1,\dots,J_n\in T_{\gamma_\tau}\Omega$ lin. indep Jacobi fields. Extend by 0 on $[\tau,\tau+h]$ to get $\overline{J_i}$, $\overline{W_j}$.

Claim: $S=\langle \overline{W_1},\dots,\overline{W_k},\overline{J_1},\dots,\overline{J_n}\rangle$ is close to a neg def subspace. Since $J_i\in\mathrm{null}(H_\tau)$,
$H(\overline{J_i},\overline{W_j})=0$
$H(\overline{J_i},\overline{J_j})=0$.

So $H|_S = \begin{array}{c|cc} & \bar W & \bar J \\ \hline \bar W & M & 0 \\ \bar J & 0 & 0 \end{array}$ where $M < 0$.
(symmetric, all eigvals < 0).
Can we perturb so that whole metric is < 0?

Let $z_i = \Delta_+ D_+ J_i$. ~~These are lin. indep.~~ For $Y \in T_{\gamma_{\tau+h}} \Omega$, $H(\bar J_i, Y) = -\langle z_i | Y(\tau) \rangle$. Since the J_i's are lin indep, so are z_i's. $\exists$ a dual basis $y_1, \dots, y_n$ s.t. $\langle y_i | z_j \rangle = \delta_{ij}$. Let $Y_i \in T_{\gamma_{\tau+h}} \Omega$ be fields s.t. $Y_i(\tau) = y_i \Rightarrow H(\bar J_i, Y_j) = \delta_{ij}$.

Let $c > 0$, let $S' = \langle \bar W_1, \dots, \bar W_k, c^{-1} \bar J_1 + cY_1, \dots, c^{-1} \bar J_n + cY_n \rangle$.
Then:
$$H|_{S'} = \left(\begin{array}{c|c} M & cH(\bar W_i, Y_j) \\ \hline & -2\delta_{ij} + c^2 H(Y_i, Y_j) \end{array} \right)$$
$H(\bar W_i, c^{-1}\bar J_j + cY_j) = H(\bar W_i, cY_j)$

$H(c^{-1}\bar J_i + cY_i, c^{-1}\bar J_j + cY_j)$
$= -2\delta_{ij} + c^2 H(Y_i, Y_j)$
$$\to \left(\begin{array}{c|c} M & 0 \\ \hline 0 & -2I \end{array} \right) \text{ as } c \to 0.$$
So if c is small, $H|_{S'} < 0 \Rightarrow \lambda(\tau + h) \geq \lambda(\tau) + n$.
If h suff small, $\lambda(\tau + h) = \lambda(\tau)$.

Therefore, λ increases by $\operatorname{ord}_{\gamma(0)} \gamma(t)$ at every conj pt $\gamma(t)$ —
$\lambda(1) = \operatorname{index} H_\gamma = \sum_{t \in (0,1)} \operatorname{ord}_{\gamma(0)} \gamma(t)$ //.

Applications: ~~Thm (Cartan-Hadamard):~~ Thm: ~~Let M be a mfld, s.t. $K(X,Y) \leq 0$ $\forall X, Y \in T_pM$.~~ $\forall p \in M$, $\forall X, Y \in T_pM$, $K(X,Y) \leq 0$ (nonpositive sectional curvature), ~~suppose that M is simply connected~~
~~If M is complete, simply connected, then $M \cong \mathbb{R}^n$, and every geodesic is length-minimizing.~~ Then every geodesic has index 0.
Pf: Let γ a geodesic, ETS no conj. pts. Let W be Jacobi, consider $\langle D_t W | W \rangle$. Then
$$\frac{d}{dt} \langle D_t W | W \rangle = \|D_t W\|^2 + \langle D_t^2 W | W \rangle$$
$$= \|D_t W\|^2 + \langle R(\gamma', W)\gamma' | W \rangle \geq 0.$$
So $\langle D_t W | W \rangle$ is nondecreasing. If $W(0) = 0$, $W(\tau) = 0$, then $\langle D_t W | W \rangle = 0 \ \forall t \in [0, \tau] \Rightarrow \|D_t W\|^2 = 0 \Rightarrow W(t) = 0 \ \forall t$ //

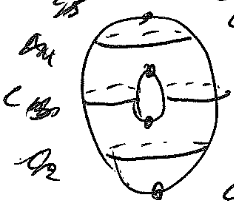
Very close to saying that every geodesic is a local min of length not quite ~~locally length-min, but not~~ quite. In fact, we can get that, but we need some topology. ~~in a nbhd~~

~~In particular, need Morse theory~~ ~~Why do we care about index?~~
~~Helps us describe topology of~~ a mfld.

Specifically:
Can we describe topology of $\Omega_{p,q} = \{$piecewise smooth paths p to $q\}$?
Yes! (1) Approximate Ω by a mfld M s.t. $E|_M$ is smooth, every geod is a crit pt of E.
(2) ~~Ther~~ Use the index of the crit pts to describe M using Morse thry.

Morse theory: Let M be a mfld, let $f: M \to \mathbb{R}$ a smooth fn.
A crit pt of f is a pt $p \in M$ s.t. $\nabla f(p) = 0$. We say p is nondegenerate if $\mathrm{null}(H_p(f)) = 0$ and write $\mathrm{index}(p) = \mathrm{index}\, H_p(f)$.

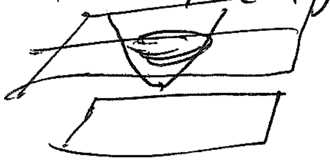
The indexes of crit pts describe M. Ex: M = torus, f = height fn:



c_4 Four crit pts, index $0, 1, 1, 2$.
c_3 Let $M^a = f^{-1}((-\infty, a])$. Then M^a changes topology
c_2 exactly at crit pts:

$a < c_1$, $a \in (c_1, c_2)$, $a \in (c_2, c_3)$, $a \in (c_3, c_4)$, $a > c_4$

Further, ~~the type~~ type of change depends on index.



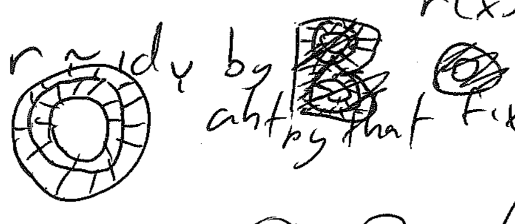
~~index 0 — disc appears~~
index 1 ~~— two parts of boundary a~~

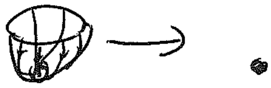
Prop: Let $a < b$, suppose that $f^{-1}([a,b])$ is compact.
– If $f^{-1}([a,b])$ contains no crit pts, then $M^a \cong M^b$ and M^b deformation retracts to M^a.
– If $f^{-1}([a,b])$ contains one critical point of index λ, then M^b def. retracts to M^a with a λ-cell attached:
That is, there is an attaching map $\alpha: \partial D^\lambda = S^{\lambda-1} \to M^a$ s.t. M^b def. retracts to $M^a \cup_\alpha D^\lambda = \dfrac{M^a \amalg D^\lambda}{p \sim \alpha(p)\ \forall p \in S^{\lambda-1}}$

overflow:

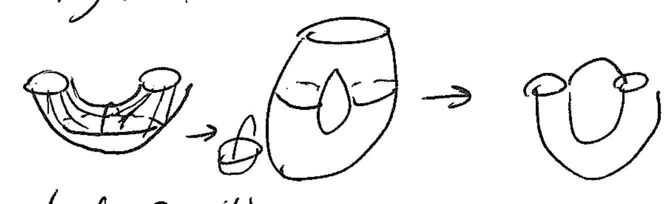
Ex: 

Def: If $X \subset Y$, X is a deformation retract of Y if ~~B~~ $\exists$ a map $r: Y \to X$ s.t. $r(x) = x\ \forall x \in X$ (r is a retraction)
and $r \simeq id_Y$ by a htpy that fixes X ptwise



Ex: 

M^a def retracts to a pt (a 0-cell)



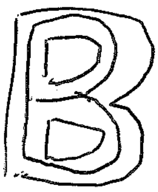
2025-04-07

Quick notes on homotopy:

Let $f,g: X \to Y$. We say f is homotopic to g if $\exists\, h: X\times[0,1]\to Y$ ($f \simeq g$) with $h(x,0)=f(x)$, $h(x,1)=g(x)$. ~~Then~~ ~~$[X,Y]$~~ ~~Then~~ (Useful in algebraic topology: can put alg. structures on $[X,Y]$ = htpy classes of maps $X\to Y$) How much topology taken? ((CW complexes?))

We say that X is homotopy equivalent to Y if $\exists\, f: X\to Y$, $g: Y\to X$ s.t. $f\circ g \simeq id_Y$, $g\circ f\simeq id_X$ — then $\forall Z$, $[X,Z]\cong[Y,Z]$, $[Z,X]\cong[Z,Y]$ - can't be distinguished by homotopy.

Simplest examples are deformation retractions: Let $A\subset X$.
A is a retract of X if $\exists\, r: X\to A$ s.t. $r(a)=a\ \forall a\in A$.
A is a deformation retraction if $\exists$ a retraction $r: X\to A$ and a homotopy $h: X\times[0,1]\to X$ s.t. $h(x,0)=x$,
$h(x,1)=r(x)$
$h(a,t)=a\ \forall a\in A, t$

Ex:  retracts to , def. retracts to 

If A is a def. retr., then $X\simeq A$.

Prop: Let $f: M\to\mathbb{R}$, let $M^a = f^{-1}((-\infty,a])$. Let $a<b$, ~~suppose that~~ suppose that $f^{-1}([a,b])$ is compact.

① - If $f^{-1}([a,b])$ contains no crit pts of f, then M^a is diffeo to M^b ~~in fact~~ and M^a is a def. retract of M^b. (state other half later)

Pf: - Gradient flow: Let $W = \frac{-\nabla f}{\|\nabla f\|^2}$. This is smooth on $f^{-1}([a,b])$ and $Wf=-1$. So for any flow line γ, $f(\gamma(t)) = f(\gamma(0)) - t$.

Let $r(x) = \begin{cases} a & \forall a\in A \\ \Phi_W^{f(x)-a}(x) & \text{for } x\notin A\end{cases}$ - this is a def retract, can be smoothed to a diffeo.

② If $f^{-1}([a,b])$ contains exactly one ~~crit~~ critical pt p and p is nondegenerate, index$(p)=\lambda$, then M^b deformation retracts to M^a with a λ-cell attached.

(i.e. there is an attaching map $\alpha: \partial D^\lambda \to M^a$ s.t. $M^b \simeq M^a \cup_\alpha D^\lambda$)

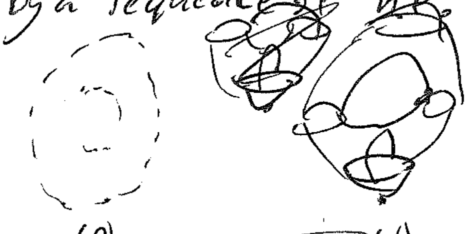
Pf: Enough to show that if $f(p)=c$, ε small, then $M^{c+\varepsilon}\simeq M^{c-\varepsilon}\cup_\alpha D^\lambda$.

Morse Lemma: $\exists$ a chart near ~~p s.t.~~ $\phi: U\to\mathbb{R}^n_\varepsilon$ near p s.t.

$$f = -((\phi^1)^2+\dots+(\phi^\lambda)^2) + (\phi^{\lambda+1})^2+\dots+(\phi^n)^2.$$

Then

~~So M is constructed by a comb of~~ So: M can be constructed by a sequence of h.e.'s and cell attachments:

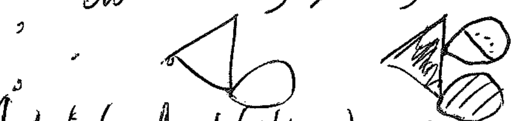


Can describe using CW complexes,

Def: A CW complex is a space constructed by inductively gluing cells: X is a ~~limit~~ union of skeleta

$X^{(0)}$ = ~~$X^{(1)}$~~ a collection of points,

$X^{(i)}$ obtained by gluing i-cells to $X^{(i-1)}$.



etc.

Thm (Whitehead-Hilton): Let X be h.e. to a CW complex Y. Let $\alpha: S^{\lambda-1}\to X$. ~~Then $X\cup_\alpha D^\lambda \simeq Y\cup_\beta D^\lambda$~~

Then $\exists$ a CW complex Y' with one more cell s.t. $X\cup_\alpha D^\lambda \simeq Y'$.

So:

Thm (Morse): If $f: M\to\mathbb{R}$ is a Morse function ($f^{-1}((-\infty,a])$ compact for all a, all crit pts nondegenerate.) then M is h.e. to a CW cplex with one λ-cell for each crit pt of index λ.

Ex: Let M be a compact manifold. If $f:M\to\mathbb{R}$ ~~has~~ is a Morse fn with two crit pts, then $M\simeq S^n$.

Pf: One pt must be a max one a min — say p = min, q = max. $a=f(p)$, $b=f(q)$. ~~By Morse~~ Then index(p) = 0, index(q) = n — So $M\simeq X$ with one 0-cell, one n-cell.

$X^{(0)}$ = • $X^{(n)} = X^{(0)}\cup_\alpha D^n$, but $\alpha:\partial D^n\to X^{(0)}$

sends ∂D^n to p. $\Rightarrow X\cong S^n$ //

In fact, M is homeo (but not diffeo) to S^n:

by Morse Lemma, ~~$f^{-1}([b-\varepsilon,b])$,~~ $f^{-1}([a,a+\varepsilon])\cong D^n\cong f^{-1}([b-\varepsilon,b])$.

Further, $f^{-1}([a+\varepsilon,b-\varepsilon])$ has no crit pts, so $f^{-1}([a,b-\varepsilon])\cong D^n$

$$M = D^n\cup_\alpha D^n \quad \alpha:\partial D^n\to\partial D^n$$

Thm (Alexander): Any such mfld is homeo to S^n:

Pf: Let (r,θ), $(r,\theta)'$ be polar coords on D, D', so that $(1,\theta)=(1,\alpha(\theta))'$. Let $(r,\theta)^N$, $(r,\theta)^S$ be polar coords on hemispheres of S^n, let

$$\beta: D\cup_\alpha D' \to S^n \quad \beta((r,\theta)) = (r,\alpha(\theta))^N$$
$$\beta((r,\theta)')=(r,\theta)^S$$

– send D' to southern diffeo

send D to northern ~~diffeo~~ as a homeo, with singularity at pole //

The path space: Recall that

$\Omega_{p,q} = \{$piecewise-smooth paths from p to $q\}$

$T_\gamma \Omega_{p,q} = \{$piecewise-smooth v. fields w/ $W(0)=0$, $W(1)=0\}$.

- Infinite-dimensional — can we approx by fin. dim?

Let $E: \Omega \to \mathbb{R}$, $E(\gamma) = \frac{1}{2}\int \|\gamma'\|^2 dt$.

$\Omega^a = E^{-1}([0,a])$, $\operatorname{Int}\Omega^a = E^{-1}([0,a))$.

For $0 = t_0 < t_1 < \cdots < t_k = 1$, let $\Omega(t_0, \cdots, t_k) = \{$broken geodesics$\}$
$= \{w \mid w|_{[t_i, t_{i+1}]}$ a geodesic$\}$.

Let $\Omega^a(t_0, \cdots, t_k) = \Omega^a \cap \Omega(t_0, \cdots, t_k)$.
$\operatorname{Int}\Omega^a(t_0, \cdots, t_k) = \operatorname{Int}\Omega^a \cap \Omega(t_0, \cdots, t_k)$.

Prop: If M is complete, then $\forall c > 0$, $\exists t_0, \cdots, t_k$ s.t. $\operatorname{Int}\Omega^c(t_0, \cdots, t_k)$ is a finite-dim manifold with tangent space $J(t_0, \cdots, t_k)$.

Pf: If $w \in \Omega^c$ then $\ell(w|_{[s,t]}) \le \sqrt{(t-s)c}$. So $w([0,1]) \subset \overline{B}_{\sqrt{c}}(p)$, which is compact. ~~Choose eps s.t. inj~~ Let $r = \inf_{q \in \overline{B}_{\sqrt{c}}(p)} \operatorname{inj\,rad}(q)$, s.t. $B_r(x)$ is a UNN $\forall x \in \overline{B}_{\sqrt{c}}(p)$

choose t_i s.t. $|t_i - t_{i+1}| < \frac{r}{c}$ - then each $w([t_i, t_{i+1}])$ is contained in a UNN. $\Rightarrow w([t_i, t_{i+1}])$ determined by endpts.
and w is determined by $w(t_1), \cdots, w(t_{k-1})$. So $\operatorname{Int}\Omega^c(t_0, \cdots, t_k)$ is homeo to an open subset of M^{k-1} //

Let $B = \Omega^c(t_0, \cdots, t_k)$, $B^a = \Omega^a(t_0, \cdots, t_k)$ $\forall a < c$. Then:

- E is smooth on B.
- $\forall a < c$, B^a is a compact def retract of Ω^a.
  (namely, given ~, retract to ⌒)
- critical pts of E on Ω^c = crit pts of $E|_B$ on B. (geodesics)
- If γ is a geodesic, $T_\gamma B = \{$broken Jacobi fields$\}$, so
  $\operatorname{index}_E(\gamma) = \operatorname{index}_{E|_B}(\gamma)$, $\operatorname{nullity}_E(\gamma) = \operatorname{nullity}_{E|_B}(\gamma)$.

So: $\Omega^a \simeq B^a$, and B^a is a fin. dim mfld, where crit pts of E are geod.

Fundamental Theorem of Morse Theory: If M is complete, p and q are not conjugate along any geod of length $\le \sqrt{a}$, then Ω^a is h.e to a CW complex with one cell of dim λ for every geodesic in Ω^a with index λ.

Last time:

Thm (Morse): Let $f: M \to \mathbb{R}$ be a Morse function on a mfld M.
($f^{-1}([-\infty, a])$ compact for all a, all critical points nondegenerate).
Then M is homotopy equivalent to a CW-complex with one λ-cell for each crit pt of index λ.

Today:

Let $\Omega = \{$piecewise-smooth paths from p to q$\}$ — apply Morse to Ω.
Let $\Omega^a = E^{-1}([0,a])$, $\operatorname{Int} \Omega^a = E^{-1}([0,a))$. For $0=t_0<t_1<\cdots<t_k=1$
let $\Omega^a(t_0,\dots,t_k) = \Omega^a \cap \{$broken geodesics$\}$
$\operatorname{Int} \Omega^a(t_0,\dots,t_k) = \operatorname{Int} \Omega^a \cap \{$broken geods$\}$.

Prop: For any $c>0$, if $|t_i - t_{i+1}|$ suff small, then
- any $w \in \Omega^c(t_0,\dots,t_k)$ is determined by $w(t_0),\dots,w(t_k)$.
- so ~~Int Ω~~ $B = \operatorname{Int} \Omega^c(t_0,\dots,t_k)$ is a finite-dim mfld. (an open subset of M^{k-1})

Pf: ~~Choose~~ If $|t_i - t_{i+1}| < \varepsilon$, then $l(w|_{[t_i,t_{i+1}]}) < \sqrt{c(t_{i+1}-t_i)}$.
- if this is small, then every segment of w lies in a UNN.

~~Thm~~ ~~- E is smooth on B~~ Let $B^a = \Omega^a(t_0,\dots,t_k)$. Then:
~~- E is smooth on B~~ ~~- B^a is compact $\forall a$~~.
- ~~B B^a~~ $\forall a$, B^a is a deformation retract of Ω^a.

So what's the topology of B^a?

- E is smooth on B - $\forall a$, B^a is compact.
- $\{$critical pts of $E|_B\} = \{$geodesics ~~of energy < c~~$\}$.
- If γ is a geodesic, then $T_\gamma B = J(t_0,\dots,t_k)$, so $\operatorname{index}_E(\gamma) = \operatorname{index}_{E|_B}(\gamma)$ and $\operatorname{nullity}_E(\gamma) = \operatorname{nullity}_{E|_B}(\gamma)$.

So $\Omega^a \simeq B^a \simeq$ CW complex. Therefore:

Fundamental Theorem of Morse Theory: If M is complete, $a>0$, $p,q \in M$ are not conjugate by any geodesic of length $\leq \sqrt{a}$, then Ω^a is h.e. to a CW complex with a ~~cell of~~ λ-cell for each geod in Ω^a with index λ.

~~Pf: $\Omega^a \simeq$~~

In fact, we can take $a=\infty$ - if p and q are not conj by any geod, then Ω is h.e. to a complex with one λ-cell ~~for~~ every geod from p to q.

Applications: Thm (Cartan-Hadamard): Let M be complete, simply-connected, $K(X,Y) \leq 0$ $\forall X,Y \in TM$, then $M \cong \mathbb{R}^n$.

Pf: Previously, M has no conjugate points, so for $p,q \in M$, every geod from p to q has index 0. $\Rightarrow \Omega_{p,q} \simeq$ union of 0-cells.

components of $\Omega_{p,q}$ = # of geodesics from p to q.

But if M is simply connected, then any two paths p to q are htpic $\Rightarrow \Omega_{p,q}$ is connected. So $\exists!$ geod from p to q.

So $\exp_p$ is injective, surjective (Hopf–Rinow), nonsingular so $\exp_p$ is a diffeomorphism //

Cor: If M complete, ~~$K(X,Y)$~~ $K_M \le 0$, then Geodesics from p to q are in 1-to-1 corresp with htpy classes of paths from p to q.

In fact, we can sketch $\Omega_{p,q}$ as

– every geodesic is the minimum-energy curve in some component of $\Omega_{p,q}$.

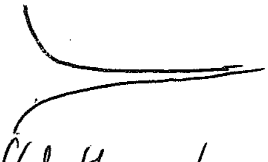
~~Another example: Minimal surfaces are important~~ (~~critical points of the area functional~~) are ~~important to geometry, but hard to find:~~

~~Defn: Def:~~ Another application: Can use this to find geodesics:

Def: A closed geodesic is a smooth ~~nonconst~~ map $\gamma: S^1 \to M$ s.t. $D_t \gamma' = 0$, eg. equator of sphere. When does a mfld contain a closed geod?

~~If~~ M is ~~complete~~ compact,

- If $\pi_1(M) \neq 0$, then M contains a closed geodesic in each nontriv ~~htpy~~ ~~htp~~ free homotopy class (htpy class of maps $S^1 \to M$).
  (can use Arzela-Ascoli to show that ~~E an energy-minimizing loop~~, E has a minimum on each class).
- If ~~$\pi_1(M) \neq 0$~~ M non compact, maybe not:



Thm (Lyusternik–Fet): If M is a compact Riem mfld, then it contains a nontriv ~~a~~ closed geodesic.

Pf: Therefore assume $\pi_1(M) = 0 \iff$ every loop $\gamma: S^1 \to M$ is htpic to a point.

Application: Topology of spheres S^n, $n \geq 2$

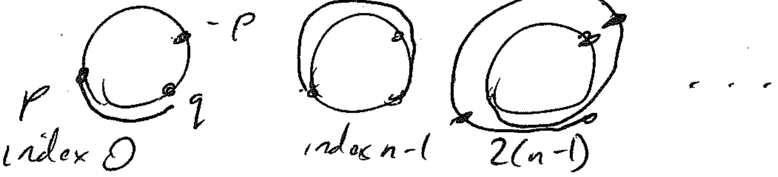
~~Let~~ Let $p \in S^n$. ~~Then p is conjugate to itself,~~ $\pm p$. ~~but~~

Then any geod of length $\pi, 3\pi$, ~~etc~~ 5π ends at $-p$,

& any geod of length $2\pi, 4\pi, 6\pi$, ends at p.

So p is conjugate to $\pm p$, with index $n-1$. ~~Therefore~~, If $q \neq \pm p$,

~~then $\Omega_{p,q}$ is h.e. to a CW complex with one cell in each dimension~~ there are ∞'ly many geodesics from p to q:



So $\Omega_{p,q}$ is h.e. to a CW complex with one cell in each dimension $0, n-1, 2(n-1)$, ...

If $n \geq 3$, then we can calculate homology of $\Omega_{p,q}$:

$$C_k(\Omega_{p,q}) = \begin{cases} \mathbb{Z} & \text{if } k = j(n-1) \\ 0 & \text{otherwise} \end{cases} \quad \text{chain complex}$$

$$\to 0 \to \cdots \to 0 \to \mathbb{Z} \to \underbrace{0 \to \cdots \to 0}_{n-2\ 0\text{'s}} \to \mathbb{Z}$$

$$H_k(\Omega_{p,q}) = \begin{cases} \mathbb{Z} & \text{if } k = j(n-1) \\ 0 & \text{otherwise} \end{cases}$$

Therefore: Thm: If $n \geq 3$ ~~and~~, M is any Riem. metric on S^n, and $p, q \in S^n$ are not conjugate, then there are ∞'ly many geodesics from p to q.

Pf: ~~$H_k(\Omega_{p,q})$ is independent of metric. So By Fund Thm~~ By FT, $\Omega_{p,q}$ is h.e. to a CW complex with one λ-cell for each geod of index λ from p to q.

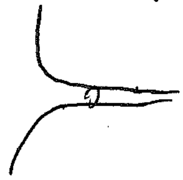
But $H_{j(n-1)}(\Omega_{p,q}) \neq 0_{\mathbb{Z}}$ $\forall j$ so there's a geod of ~~length~~ index $j(n-1)$ $\forall j$. //

In fact, we can ~~do better~~ use this to study more closely ...

Application: Closed geodesics.

Def: A closed geodesic is a smooth map $\gamma : S^1 \to M$ s.t. $D_t \gamma' = 0$

eg. equator of sphere. When does a mfld contain a nontrivial closed geod?

Ex: $M =$  - non compact, no closed geodesics.

Thm (Lyusternik-Fet): If M is a ~~closed~~ compact Riemannian manifold, then it contains a nontrivial closed geodesic.

Pf: If $\pi_1(M) \neq 0$, then M contains a closed geodesic in every free homotopy class (homotopy class of maps $S^1 \to M$ — these are in bij corresp to the conjugacy classes of π_1)

By Arzela-Ascoli and compactness, ~~energy~~ minimum energy achieved by some curve, this curve is a geodesic.

Claim: If so, then $\Lambda M \simeq M$.

So suppose $\pi_1(M) = 0$ and M has no closed geodesics.
Do Morse theory on $\Lambda M = \{\text{free loop space}\}$
$= \{\text{piecewise smooth } \gamma : S^1 \to M\}$

As before, $\Lambda^a = E^{-1}((-\infty, a])$, $\Lambda(t_0, \dots, t_k) = \{\text{broken geodesics}\}$ etc.
If $|t_i - t_{i+1}|$ suff small, then $\Lambda^a(t_0, \dots, t_k) = C^a$ is a mfld, critical pts of energy are closed geodesics, etc.

But! Trivial closed geodesics (constants) are degenerate crit pts:
$E^{-1}(0) = M \subset \Lambda M$.

~~Bf~~ and if $\varepsilon > 0$ suff small, then $C^\varepsilon \simeq M$

So we can do Morse, but instead of starting with $\emptyset$, start with M:

$\Lambda M \simeq M$, with one λ-cell attached for each nontriv closed geod of index λ
$= M$ // - In fact, there is a def. retract from ΛM to M.

Claim: This leads to a contradiction.

I need one topological fact: If M is a compact mfld, then $\exists k > 0$ s.t. $\pi_k(M) = [S^k, M] \neq 0$. Let k_0 be the smallest such k, let $\alpha : S^{k_0} \to M$ be homotopically nontrivial.

~~We can write $S^{k_0} \simeq [0,1]^{k_0} / \partial [0,1]^{k_0}$~~ - let $D^{k_0} = [0,1]^{k_0}$, we can write
$S^{k_0} = D^{k_0} / \partial D^{k_0}$, ~~write~~ so write $* = \alpha(\partial D^{k_0})$.

So

α is a map . ~~Let~~ Write $D^{k_0} = D^{k_0-1} \times [0,1]$

Let $\bar{\alpha} : D^{k_0-1}/\partial D^{k_0-1} \to \Lambda M$, $\bar{\alpha}(x)(t) = \alpha(x,t)$ –

each $\bar{\alpha}(x)$ is a loop based at $*$, if $x \in \partial D^{k_0}$, then $\bar{\alpha}(x) = [*]$.
So $\bar{\alpha} : S^{k_0-1} \to \Lambda M$. By the retract, $\bar{\alpha}$ is htpic to ~~$\bar{\beta}$ where~~
$\bar{\beta} : S^{k_0-1} \to M$, ~~i.e.~~ ~~$\bar{\beta}$ constant~~ By minimality,
$\bar{\beta}$ is null htpic. Therefore α is null-homotopic $\divideontimes$.

Overflow: explain in terms of sweepouts, mean curvature flow.