2025-03-05

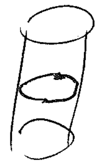
Last time: Hopf-Rinow:
If $M$ is connected and geodesically complete, then $\forall x,y \in M$, $\exists$ a length-minimizing geodesic from $x$ to $y$.

Cor: If $M$ is geodesically complete and $B_r = \{y \in M \mid d(x,y) < r\}$, $\overline{B_r}(x) = \{y \in M \mid d(x,y) \leq r\}$, then $B_r(x) = \exp_x B_r(0)$, $\overline{B_r}(x) = \exp_x \overline{B_r}(0)$.

In particular, $\overline{B_r}(x)$ is compact.

Cor: $M$ is complete (Cauchy sequences converge) $\Leftrightarrow$ $M$ is geodesically complete.
Pf: Exercise

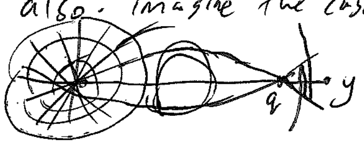
~~Today~~ Next: Deeper into: when is a geodesic minimal/not minimal?
We know: As long as $l(\gamma) <$ injrad$(0)$, $\gamma$ is minimizing.
When $l(\gamma) \geq$ injrad, can construct counterexamples:



cylinder, $\gamma$ goes halfway around.

More generally, if $M$ has a geodesic loop, ~~then~~

But also: imagine the case where geodesics curve:




- exp has a singularity at $q$.
- three geods from $x$ to $y$.
- one of them probably isn't minimal.

How can we analyze this?

Calculus of Variations:
For $p,q \in M$, let $\Omega_{p,q} = \{$piecewise-smooth paths from $p$ to $q\}$ with the $C^1$ topology: $\gamma$ is close to $\lambda$ if $\gamma(t)$ is close to $\lambda(t)$, $\gamma'(t)$ close to $\lambda'(t)$. Then $l: \Omega_{p,q} \to \mathbb{R}$ is continuous.
We want to treat this like a manifold.
~~Let~~ Let the "tangent space" $T_\gamma \Omega = \{$piecewise-smooth vector fields on $\gamma$ with $V(0) = V(1) = 0\}$

A variation of $\gamma$ is a path $\bar{\alpha}: (-\varepsilon, \varepsilon) \to \Omega_{p,q}$ s.t.
① $\bar{\alpha}(0) = \gamma$. ② $\exists$ partition $0 = t_0 < t_1 < \cdots < t_n = 1$ s.t. $\alpha(u,t) = \bar{\alpha}_u(t)$, is smooth on each $(-\varepsilon,\varepsilon) \times [t_i, t_{i+1}]$.
Then $\frac{\partial \alpha}{\partial u}$ is defined and $\frac{\partial \alpha}{\partial u}\big|_{u=0} \in T_\gamma \Omega$.

Conversely, $\forall\, W \in T_\gamma \Omega$, $\exists$ a variation $\bar{\alpha}$ with $\frac{\partial \alpha}{\partial u} = W$.

Let $E : \Omega_{p,q} \to \mathbb{R}$, $E(\gamma) = \frac{1}{2}\int_0^1 \|\gamma'(t)\|^2 dt$. Then:

Thm: $\gamma$ is a geodesic $\iff$ $\gamma$ is a critical point of $E$ (i.e., $\frac{d}{dt} E(\bar{\alpha}_u) = 0$ for any variation)

Pf: First Variation Formula: Let $\bar{\alpha}$ be a variation of $\gamma$, ~~...~~ discont at $t_i$'s.
Calculate $\frac{d}{du}[E(\bar{\alpha}_u)]_{u=0}$. $E(\bar{\alpha}_u) = \frac{1}{2}\int_0^1 \|\frac{\partial \alpha}{\partial t}\|^2 dt = \frac{1}{2}\int_0^1 \langle \frac{\partial\alpha}{\partial t} | \frac{\partial\alpha}{\partial t}\rangle dt$

$$\frac{d}{du}[E(\bar{\alpha}_u)] = \frac{1}{2}\int_0^1 \frac{d}{du}\langle \frac{\partial \alpha}{\partial t} | \frac{\partial\alpha}{\partial t}\rangle dt \qquad \text{Let } \partial_u = \frac{\partial\alpha}{\partial u},\ \partial_t = \frac{\partial\alpha}{\partial t}$$

$$= \cancel{\langle \frac{\partial^2\alpha}{\partial u \partial t} | \frac{\partial\alpha}{\partial t}\rangle} \int_0^1 \langle D_u \frac{\partial\alpha}{\partial t} | \frac{\partial\alpha}{\partial t}\rangle dt$$

Torsion trick: $\tau(\frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t}) = 0 = D_u\partial_t - D_t\partial_u$

$$\frac{d}{du}[E(\bar{\alpha}_u)] = \int_0^1 \langle D_t \partial_u | \partial_t\rangle dt. \quad \text{Integ. by parts:}$$

$$\langle D_t\partial_u | \partial_t\rangle = \partial_t\langle \partial_u | \partial_t\rangle - \langle \partial_u | D_t\partial_t\rangle,$$

but $\partial_t$ is discontinuous at the $t_i$'s. So

$$\int_0^1 \langle D_t\partial_u | \partial_t\rangle dt = \sum_i \int_{t_i}^{t_{i+1}} \partial_t \langle \partial_u | \partial_t\rangle dt - \int_0^1 \langle \partial_u | D_t\partial_t\rangle$$

$$= \sum_i \langle \partial_u | \partial_t\rangle\Big|_{t_i^+}^{t_{i+1}^-} - \int_0^1 \langle \partial_u | D_t \partial_t\rangle.$$

Let $W = \partial_u$ = variation field, $V = \partial_t$ = velocity, ~~$A = D_t V$~~.
Write $V(t_i^\pm) = \lim_{t \to t_i^\pm} V(t)$, $\underbrace{\Delta_{t_i} V = V(t_i^+) - V(t_i^-),\ A = D_t V.}_{\text{two measures of acceleration}}$

Then $\frac{d}{du} E(\bar{\alpha}_u) = \sum_i \left[\langle W(t_{i+1}) | V(t_{i+1}^-)\rangle - \langle W(t_i) | V(t_i^+)\rangle\right] - \int_0^1 \langle W | A\rangle dt$

$$= \sum_i \langle W(t_i) | -\Delta_{t_i} V\rangle - \int_0^1 \langle W | A\rangle dt = -\sum_t \langle W | \Delta_t V\rangle - \int_0^1 \langle W | \underset{D_tV}{A}\rangle dt$$

~~" $= -\int_0^1 \langle W | dK(t)\rangle$" where $K$ is~~ This is the FVF — it says that $E(\bar{\alpha}_u)$ is "$C^1$" continuously differentiable and gives a formula for derivative.

Pf of Thm: If $\gamma$ is a geodesic, then $D_t V = 0$, $\Delta_t V = 0 \Rightarrow \frac{d}{du} E(\bar{\alpha}_u) = 0$.
Conversely, if $\gamma$ is a crit pt of energy, then
$$-\int_0^1 \langle W | D_t V\rangle dt - \sum_t \langle W | \Delta_t V\rangle = 0 \quad \forall W \Rightarrow D_t V = \Delta_t V = 0;$$

so $\gamma'$ is continuous, $D_t \gamma' = 0 \Rightarrow \gamma$ is a geod //

What sort of critical point is it?

Second Variation Formula: If $\gamma$ is a geodesic, $W_1, W_2 \in T_\gamma \Omega$, and $\alpha : (-\varepsilon, \varepsilon)^2 \to \Omega$ is s.t. $\alpha(0,0) = \gamma$, $\frac{\partial \alpha}{\partial u_i} = W_i$, then

$E(\alpha_{u_1, u_2})$ is smooth and ~~$H(E)(W_1, W_2) = \frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0) = -\sum \langle W_2 | \Delta_t D_t W_1 \rangle + \int_0^1 \langle W_2 | D_t^2 W_1 - R(V, W_1)V \rangle$~~

$$H(E)(W_1, W_2) = \frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0) = -\sum_t \langle W_2 | \Delta_t D_t W_1 \rangle - \int_0^1 \langle W_2 | D_t^2 W_1 - R(V, W_1)V \rangle dt$$

where $\Delta_t D_t W_1(t) = D_t W_1(t^+) - D_t W_1(t^-)$.

Pf: Differentiate $\frac{\partial E}{\partial u_2} = -\sum_t \langle \partial_{u_2} | \Delta_t \partial_t \rangle - \int_0^1 \langle \partial_{u_2} | D_t \partial_t \rangle dt$.

$$\frac{\partial^2 E}{\partial u_1 \partial u_2} = -\sum_t \langle D_{u_1} \partial_{u_2} | \cancel{\Delta_t \partial_t} \rangle_{\to 0} - \sum_t \langle \partial_{u_2} | D_{u_1} \Delta_t \partial_t \rangle$$

$$- \int_0^1 \langle D_{u_1} \partial_{u_2} | \cancel{D_t \partial_t} \rangle_{\to 0} - \langle \partial_{u_2} | D_{u_1} D_t \partial_t \rangle dt$$

At $(0,0)$, $\frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0) = -\sum_t \langle W_2 | \underset{\text{(linearity)}}{D_{u_1} \Delta_t} \frac{\partial \alpha}{\partial t} \rangle - \int_0^1 \langle W_2 | \underset{\text{(def of R)}}{D_{u_1} D_t} \frac{\partial \alpha}{\partial t} \rangle$

$$= -\sum_t \langle W_2 | \Delta_t \underset{\text{(torsion trick)}}{D_{u_1} \frac{\partial \alpha}{\partial t}} \rangle - \int_0^1 \langle W_2 | \underset{\text{(torsion)}}{D_t D_{u_1}} \frac{\partial \alpha}{\partial t} - R(V, W_1)V \rangle$$

$$= -\sum_t \langle W_2 | \Delta_t \underset{\text{(torsion trick)}}{D_t \frac{\partial \alpha}{\partial u_1}} \rangle - \int_0^1 \langle W_2 | D_t D_t \frac{\partial \alpha}{\partial u_1} - R(V, W_1) V \rangle$$

$$= -\sum_t \langle W_2 | \Delta_t D_t W_1 \rangle - \int_0^1 \langle V_2 | D_t^2 W_1 - R(V, W_1) V \rangle \,//$$

Note: Since $E$ is smooth, this has to be symmetric in $V_1, V_2$ — it is, but that's not obvious.

How do we study this? Remember the theory in $\mathbb{R}^n$ — if $f : \mathbb{R}^n \to \mathbb{R}$, $D_0 f = 0$, then $H(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & & \\ & \ddots & \\ & & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$ is a symmetric matrix, s.t. if $V_1, V_2 \in \mathbb{R}^n$ then

$$V_1 V_2 f(0) = \frac{\partial^2}{\partial u_1 \partial u_2} f(u_1 V_1 + u_2 V_2) = V_1^T H(f) V_2$$

So if $Q(V) = \frac{1}{2} V^T H_f V = \frac{1}{2} V^2 f(0,0)$, then

~~$f(x) = Q(x) + \|x\|^3$.~~

Alt: So if $\beta : (-\varepsilon, \varepsilon) \to \Omega$ is a variation with $\frac{\partial \beta}{\partial u} = W$, then

$\frac{d}{du} E(\beta_u) = 0$, $\frac{d^2}{du^2} E(\beta_u) = H(E)(W, W) \Rightarrow$

$$E(\beta_u) = \frac{1}{2} H(E)(W, W) u^2 + O(u^3) = \frac{1}{2} \langle W | D_t^2 W - R(V, W)V \rangle.$$

This is a quadratic form on $T_\gamma \Omega$ — how does it behave?

2025-03-10

Last time: Second Variation Formula.
If $\gamma$ is a geodesic, $W_1, W_2 \in T_\gamma\Omega$, $\bar\alpha : (-\varepsilon,\varepsilon)^2 \to \Omega$ is a 2-parameter variation such that $\bar\alpha_{(0,0)} = \gamma$, $\frac{\partial\bar\alpha}{\partial u_i}\Big|_{(0,0)} = W_i$, ~~then~~ $V = \gamma'$, then

$$\frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0) = -\sum_t \langle W_2 | \Delta_t D_t W_1\rangle - \int_0^1 \langle W_2 | D_t^2 W_1 - R(V,W_1)V\rangle\, dt$$

Also call this $H(E)(W_1, W_2)$ (Hessian of the energy functional)

If $\bar\beta : (-\varepsilon,\varepsilon) \to \Omega$ is a variation of ~~geodesic~~ $\gamma$ w/ $\frac{\partial\beta}{\partial u} = W$, then $\frac{d}{du}E(\bar\beta_u) = 0$, $\frac{d^2}{du^2}E(\bar\beta_u) = H(E)(W,W)$.

$$E(\bar\beta_u) = \tfrac{1}{2} H(E)(W,W)\, u^2 + O(u^3)$$

How does $H(E)(W,W)$ behave?

$H(E)$ is symmetric bilinear form, so $Q(W) = H(E)(W,W)$ is a quadratic form. ~~In $\mathbb{R}^n$, same thing: if $f$ is symmetric, bilinear, then $Q(v) = f(v,v)$ is a quadratic form.~~ So recall how these work in $\mathbb{R}^n$: if $f(v,w)$ is symmetric bilinear, then $f(v^i\partial_i, w^j\partial_j) = v^i w^j M_{ij}$ — or $f(v,w) = v^T M w$ for some matrix. If $f$ symmetric, so is $M$, and $Q(v) = v^T M v$ is a quadratic form.
Let $e_1, \dots, e_n \in \mathbb{R}^n$ an orthonormal basis of eigvecs, $Me_i = \lambda_i e_i$.
— then

$$Q(x^i e_i) = (x^i e^i)^T M (x^j e^j) = \sum_i \lambda_i (x^i)^2$$

That is, ~~up to rotation~~ shape $Q$ depends on the eigvecs of $M$.

When $W$ is smooth, $H(E)(W,W) = \int_0^1 \langle W | -D_t^2 W + R(V,W)V\rangle\, dt$

$$= \langle W | M(W)\rangle$$

where $M(W) = -D_t^2 W + R(V,W)V$ is a linear operator on $T_\gamma\Omega$.

Typically, some positive eigvals, some negative, some zero — today, we're interested in the zero eigvals.

**Jacobi fields**: If $W \in \mathcal{V}(\gamma)$ is smooth, $\gamma$ is a geodesic, $V = \gamma'$, $D_t^2 W = R(V,W)V$, then we call $W$ a Jacobi field. (Note: $W$ need not be zero at the endpoints)

Thm: $W$ is a Jacobi field if and only if $W$ is the variation field of a variation through geodesics: (a variation $\bar\alpha : (-\varepsilon,\varepsilon) \times [0,1] \to M$ s.t. $\bar\alpha_u$ is a geodesic for all $u$).

Pf: ($\Leftarrow$) Suppose $\bar{\alpha}_u$ is a geod for all $u$. ~~Then the Claim: $\frac{\partial \alpha}{\partial u}$ is Jacobi~~: Then:

$$D_t^2 \frac{\partial \alpha}{\partial u} = D_t D_t \frac{\partial \alpha}{\partial u} \xrightarrow{\text{torsion trick}} = D_t D_u \frac{\partial \alpha}{\partial t} = \cancel{D_u D_t \frac{\partial \alpha}{\partial t}}_0 + R\left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial u}\right)\frac{\partial \alpha}{\partial t}$$

$$D_t^2 W = R(V,W)V \Rightarrow W \text{ is Jacobi.}$$

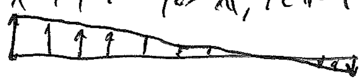
($\Rightarrow$) $D_t^2 W = R(V,W)V$ is a 2nd-order system, so solutions are determined by $W(0)$, $D_t W(0)$. $\Rightarrow$ dim{Jacobi fields} $= 2n$.

Claim: dim{variation fields of vars through geods} $= 2n$.

Let $U$ be a UNN of $\gamma(0)$, let $\varepsilon > 0$ s.t. $\gamma(\varepsilon) \in U$.
For $v_1 \in T_{\gamma(0)}M$, $v_2 \in T_{\gamma(\varepsilon)}M$, let $\lambda_1, \lambda_2$ be curves s.t. ~~$\lambda(0) = \gamma(0)$~~, $\lambda_i'(0) = v_i$. Let $\bar{\alpha}_u = \gamma_{\lambda_1(u), \lambda_2(u)}$ = geod s.t. $\bar{\alpha}_u(0) = \lambda_1(u)$, $\bar{\alpha}_u(\varepsilon) = \lambda_2(u)$.
If $u$ is suff small, this extends to $[0,1]$ and depends smoothly on $u$.
So $\frac{\partial \alpha}{\partial u}$ is a ~~variation~~ Jacobi field with $\frac{\partial \alpha}{\partial u}(0) = \lambda_1' = v_1$,
$\frac{\partial \alpha}{\partial u}(\varepsilon) = \lambda_2' = v_2$

~~That is, this is an injective map $T_{\gamma(0)}M \times T_{\gamma(\varepsilon)}M \to$ {Jacobi fields}.~~
That is, the natural map {Jacobi fields} $\to T_{\gamma(0)}M \times T_{\gamma(\varepsilon)}M$ is surjective. Both dim $2n$, so it's an isomorphism, ~~so~~ every Jacobi field arises this way. //

This link is powerful - you can describe nearby geodesics using Jacobi fields.
Ex: $M = \mathbb{R}^n$, $W \in \mathcal{V}(\gamma)$ Jacobi $\Leftrightarrow D_t^2 W = \cancel{R(V,W)V}_0 \Rightarrow W''(t) = 0 \ \forall t$
$\Rightarrow W = X + tY$ for $X, Y \in \mathbb{R}^n$.
So:



Move the endpts of a geodesic, the change in the geodesic is Jacobi = linear

E: $M = S^2$. ① Calculate $R$: ~~$K(X,Y)$~~ if $X, Y$ o.n., then

$$K(X,Y) = \langle R(X,Y)Y \mid X\rangle = 1 \quad \Rightarrow \quad R(X,Y)Y = X$$
$$\langle R(X,Y)Y \mid Y\rangle = 0 \qquad\quad R(X,Y)X = -Y.$$

So if $\gamma$ is equator, $N$ = northward, $E$ = eastward $= \gamma'$ and $W = fN + gE$, then $N, E$ are parallel, so

$$D_t^2 W = f''N + g''E$$

$$R(V,W)V = R(E, fN + gE)E = fR(E,N)E = -fN.$$

~~$D_t^2 W = R(V,W)V$~~ $W$ Jacobi $\Leftrightarrow f'' = -f \Rightarrow f = c\cos(t) + d\sin(t)$
$g'' = 0 \qquad g = a + bt.$

If $f = 0$, $W = (a + bt)E$ — tangential Jacobi field, corresponds to reparameterizing $\gamma$.
If $g = 0$, $W = (c\cos(t) + d\sin(t))N$ is orthogonal to $\gamma$ — normal Jacobi field

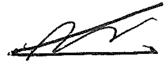
- any normal perturbation of a geodesic repeats with period $2\pi$.

Properties: (1) The Jacobi fields form a $2n$-dim subsp of $\mathcal{V}(\gamma)$

(2) Any Jacobi field decomposes as $J = J^\perp + J^\parallel$ where $J^\perp, J^\parallel$ are Jacobi, $J^\perp$ is normal to $\gamma$, $J^\parallel$ is parallel, and $J^\parallel(t) = (a+bt)\gamma'$.
(Exercise)

(3) If $p \in M$, $V, W \in T_pM$, $\gamma(t) = \exp_p(tV)$, and $J \in \mathcal{V}(\gamma)$ is a Jacobi field with $J(0)=0$, $D_tJ(0) = W$, then

$$J(1) = (D\exp_p)_V(W) = \frac{d}{du}\left[\exp_p(V+uW)\right]\Big|_0.$$



Pf: Let $\alpha_u(t) = \exp_p(V+uW)t$ - this is a variation through geodesics ~~with one endpt fixed~~, ~~Let $K = \frac{\partial\alpha}{\partial u}|_{u=0}$~~ ~~- this is a Jacobi field.~~ Let $K = \frac{\partial\alpha}{\partial u}\big|_{u=0}$. Then $K$ is Jacobi and

$$K(t) = \frac{d}{du}\Big|_{u=0}\exp_p(Vt + utW) = (D\exp_p)_{Vt}(tW)$$

Further, $K(0)=0$, $D_tK(0) = D_t\frac{\partial\alpha}{\partial u}\Big|_{(0,0)} = D_u\frac{\partial\alpha}{\partial t}\Big|_{(0,0)} = D_u[V+uW]\Big|_{(0,0)} = W$.

So $J = K$, $J(1) = (D\exp)_V(W)$ //

Ex: Polar coords on $\mathbb{R}^2$: Let $V_\theta = (\cos\theta, \sin\theta)$, $N_\theta = (-\sin\theta, \cos\theta)$, consider

$\alpha(r,\theta) = \exp_0(rV_\theta)$. Then $\alpha$ is a variation through geodesics, $J = \frac{\partial\alpha}{\partial\theta}\big|_{\theta=0}$ is a Jacobi field. By above,
$J(0)=0$, $D_tJ(0) = \frac{d}{d\theta}V_\theta = N_\theta$ - but we solved this equation,

$\frac{\partial\alpha}{\partial\theta} = J(t) = tN_\theta$. So if we think of $\alpha$ as a chart, $\frac{d\alpha}{dr} = V_\theta$, $\frac{\partial\alpha}{\partial\theta} = tN_\theta$, are orthogonal so we can write the metric as $dg^2 = dr^2 + r^2 d\theta^2$.

Likewise on the sphere, $\alpha(r,\theta) = \exp_p(rV_\theta)$, $J = \frac{\partial\alpha}{\partial\theta}$, $J(0)=0$, $D_tJ(0) = N_\theta$. Since $J(0) \perp \gamma'(0)$ & $D_tJ(0) \perp \gamma'(0)$, $J^\parallel = 0$, ~~$J = J^\perp$~~, so $J$ is orthogonal to $\gamma$, By Jacobi eq,
$J(r) = \sin(r)\cdot N$, ~~for some normal field~~ where $N$ is o.n to $\gamma'$.
so metric is $dg^2 = dr^2 + (\sin r)^2 d\theta^2$

2025-03-12

Last time: Jacobi fields:

~~① A field $J \in V(\gamma)$ is Jacobi if $D_t^2 J = 1$~~

Let $\gamma$ a geodesic, $V = \gamma'$. A field $J \in V(\gamma)$ is Jacobi if
$D_t^2 J = R(V, J)V$, $\forall t$.

① These form a $2n$-dim subset of $V(\gamma)$. ($J$ determined by $J(0)$, $D_t J(0)$)

② $J$ is Jacobi $\iff$ $J$ is variation field of a variation through geodesics.

~~Today: Use for minimizing geodesics.~~

③ We can use Jacobi fields to study $D\exp_p$ — let $p \in M$, $V, W \in T_pM$. ~~for $u \in (-\varepsilon, \varepsilon)$, let~~ let $\gamma_V(t) = \exp(tV)$ be geodesic in dir of $V$.

let $\alpha$ be variation $\Gamma_u(t) = \gamma_{V+uW}(t) = \exp_p(t(V+uW))$

~~This is a variation through geodesics, so $J = \frac{\partial\alpha}{\partial u}$ is Jacobi~~

~~(t) the Jacobi field w/ $J(0) = \frac{d}{du}[\alpha_u(0)]_{u=0} = 0$~~

~~$D_t J(0) = D_t \frac{\partial\alpha}{\partial u}(0,0) = D_u \frac{\partial\alpha}{\partial t}(0,0) = D_u[\overline{\alpha}_u'](0)$~~

~~$= \frac{d}{du}[V + uW] = W$~~

Then. on one hand, $\frac{\partial\alpha}{\partial u}\big|_{u=0} = \frac{\partial}{\partial u}\big|_{u=0} \exp_p(t(V+uW))$
$= (D\exp_p)_{tV}(tW)$. OTOH

$\alpha$ is a var through geods — so $\frac{\partial\alpha}{\partial u}\big|_{u=0}$ is Jacobi. Specifically, if $J = \frac{\partial\alpha}{\partial u}\big|_{u=0}$, then $J$ is the Jacobi field with

$J(0) = (D\exp_p)_0(0) = 0$

$D_t J(0) = D_t \frac{\partial\alpha}{\partial u}\big|_{(0,0)} = D_u \frac{\partial\alpha}{\partial t}\big|_{(0,0)} = \frac{d}{du}\{\overline{\alpha}_u'(0)\}\big|_{u=0} = \frac{d}{du}(V+uW)$
$= W$.

~~Useful to define: In part~~

Prop: If $p \in M$, $V, W \in T_pM$, $J$ is a Jacobi field on $\gamma_V$ with $J(0) = 0$, $D_t J(0) = W$, then $(D\exp_p)_V(W) = J(1)$.

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Today: Use to study minimizing geodesics:

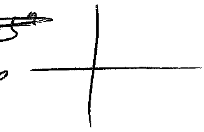
Def: If $\gamma: \mathbb{R} \to M$ is a geodesic, $\gamma(0) = p$, $\gamma(t) = q$, we say that $q$ is <u>conjugate</u> to $p$ along $\gamma$ if $\exists$ a Jacobi field $W$ on $\gamma$ s.t. $W(p) = 0$, $W(q) = 0$, but $W \neq 0$. The <u>order of conjugacy</u> of $q$ is $\mathrm{ord}_p(q) = \dim\{W \in V(\gamma) \mid W \text{ is Jacobi}, W(p) = 0, W(q) = 0\}$.

- If $p$ and $q$ are not conjugate along $\gamma$, then any Jacobi field is determined by $W(p)$, $W(q)$ (because $W \mapsto (W(p), W(q))$ is a bij.
- If $p$ and $q$ are conjugate, ~~then along then~~ along some geod $\gamma_V$, ~~ord~~ $q$ is a singular point of $\exp_p$ ~~with~~ — indeed, ~~$D\exp_p(tV)$~~ $(D\exp_p)_{tV}$ has rank $n - \mathrm{ord}_p(q)$.
- So conjugate pts are rare — set of $q$ s.t. $p$ conj to $q$ along some geod has measure zero.

Ex: $M = S^2$, $\gamma(0) = p$, $\gamma$ unit speed. ~~Can show (ex) that X~~ Let $V = \gamma'(t)$, $N$ = a unit field to $V$.
Last time, calculated that Jacobi fields are like

$$J(t) = (a + bt)V + (c\cos(t) + d\sin(t))N.$$

So if $J(0) = 0$ ~~& $J(t) = 0$, $t \neq 0$, then $a = b = 0$~~,
then $a = c = 0$. $J(t) = btV + d\sin(t)N$, so $J(t) = 0 \Rightarrow b = 0$, $t = k\pi$. — $\gamma$ has conj pts spaced $\pi$ apart.

Corresponds to singularities of exp: $\exp_p$  sends circle of radius $k\pi$ to either $p$ or $-p$. — singularity.
More generally, if $M = S^n$, then $\gamma(0)$ is conj to $\gamma(k\pi)$ with order $n-1$.

Note: Before first conj pt, $\gamma$ is length-minimizing — after, its not.
This is a general phenom:

Thm If $\gamma$ is a length-minimizing geodesic from $p$ to $q$, then the interior of $\gamma$ contains no points conjugate to $p$.

Pf: Suppose $\gamma(0) = p$, $\gamma(1) = q$. Suppose $t_0 \in (0,1)$, $p$ conj to $\gamma(t_0)$. Claim: $\exists\, Z \in T_\gamma\Omega$ st. $H(E)(Z,Z) < 0$.

Let $X$ a nonzero Jacobi field s.t. $X(0) = 0$, $X(t_0) = 0$.
Let $Y(t) = \begin{cases} X(t) & 0 \le t \le t_0 \\ 0 & t > t_0 \end{cases}$ Then, by 2VF, for any $W \in T_\gamma\Omega$,

$$H(W,Y) = -\sum_t \langle W | \Delta_t D_t Y\rangle - \int_0^1 \langle W | D_t^2 Y - R(\gamma', Y)\gamma'\rangle dt$$

$$= -\langle W(t_0) | \Delta_t D_t Y(t_0)\rangle$$

$$= \langle W(t_0) | D_t X(t_0)\rangle. \quad \text{Since } X \not\equiv 0,\ D_t X(t_0) \neq 0.$$

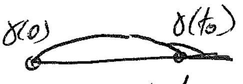
just depends on $W(t_0)$.
So: Let $U \in T_\gamma\Omega$ a smooth field w/ $U(t_0) = -D_t X(t_0)$,
let $Z = Y + \varepsilon U$. Then

$$H(Z,Z) = H(Y,Y) + 2\varepsilon H(Y,U) + \varepsilon^2 H(U,U)$$

$$= 0 \cancel{+} - 2\varepsilon\langle U(t_0) | D_t X(t_0)\rangle + \varepsilon^2 H(U,U)$$

$$= -2\varepsilon \|D_t X(t_0)\|^2 + \varepsilon^2 H(U,U) < 0 \text{ when } \varepsilon \text{ small.}$$

So $\gamma$ is not a local minimum. //

A little odd, but here's a pic: 

doesn't change length much, but creates a corner

(Try this on $S^2$).

Further, curvature implies existence of conj pts.

Def: $\forall p \in M$, if $E_1, \dots, E_n \in T_pM$ is an o.n basis, $U_1, U_2 \in T_pM$, let

$$Ric(U_1, U_2) = \sum \langle R(E_i, U_1)U_2 | E_i \rangle.$$

This is the <u>Ricci curvature</u> of $M$, a symm. bilinear form.

In particular, if $X$ is a unit vector, we can take $X = E_1, E_2, \dots, E_n$ a o.n basis, then

$$Ric(X,X) = \sum \langle R(E_i, X)X | E_i \rangle = \sum_{k=2}^{n} K(X, E_k).$$

– measures "average sectional curvature".

Thm (Bonnet-Myers): Suppose $M$ is an $n$-manifold $r > 0$, and $\forall U \in TM$ ~~s.t. $\|U\|=1$,~~ if $\|U\| = 1$, then $Ric(U,U) \geq \frac{n-1}{r^2}$. (for instance, if $K(X,Y) \geq \frac{1}{r^2}\ \forall X, Y$). Then any geodesic of length $\geq \pi r$ is not minimizing.
~~In particular, if $M$ is complete, then $diam(M) \leq \pi r$.~~

Pf: Let $\gamma : [0,1] \to M$ a geodesic of length $L > \pi r$, let $E_0, \dots, E_{n-1}$ parallel orthonormal fields on $\gamma$ s.t. $E_0 = \frac{\gamma'}{\|\gamma'\|}$. Let $W_i = \sin(\pi t)E_i$.

Claim: $H(W_i, W_i) < 0$ for some $i$:

We have:

$$H(W_i, W_i) \cancel{= \int_0^1 \pi^2 \sin^2(\pi t)}$$
$$\cancel{= -\int_0^1 \langle W_i | R(\gamma', W_i)\gamma' \rangle dt}$$
$$\cancel{= \int_0^1 \sin^2(\pi t) \langle R(E_i, \gamma')\gamma' | E_i \rangle dt}$$
So $\cancel{\sum_i H(W_i, W_i) = \sum_i \int_0^1 \sin^2(\pi t) L^2\, Ric(\gamma', \gamma')\, K(E_i, \gamma') dt}$
$\cancel{= \int}$

$$H(W_i, W_i) = -\int_0^1 \langle W_i | \underbrace{D_t^2 W_i}_{-\pi^2 \sin(\pi t) E_i} - R(\gamma', W_i)\gamma' \rangle dt$$
$$= \int_0^1 \pi^2 \sin^2(\pi t) dt + \int_0^1 \langle W_i | R(\gamma', W_i)\gamma' \rangle$$
$$= \frac{\pi^2}{2} + \int_0^1 \sin^2(\pi t) L^2 \langle R(E_1, E_i)E_1 | E_i \rangle dt$$
$$= \frac{\pi^2}{2} - \int_0^1 L^2 \sin^2(\pi t) K(E_1, E_i) dt.$$
$$\sum_{i=2}^{m} H(W_i, W_i) = (m-1)\frac{\pi^2}{2} - \int_0^1 L^2 \sin^2(\pi t) Ric(E_1, E_1) dt$$
$$\leq (m-1)\frac{\pi^2}{2} - L^2 \frac{1}{2} \cdot \frac{m-1}{r^2} \quad \text{– if } L > \pi r.$$

This is $< 0$. $\Rightarrow$ $\gamma$ is nonminimizing.

Cor: If $M$ is complete, then $diam(M) \leq \pi r$ //