Differential Geometry II — 2025-01-22

Robert Young (ryoung@cims.nyu.edu)
Course site: cims.nyu.edu/~ryoung/courses/diffgeo
Office hrs: Mondays, 10:30-11:30.
Announcements through Brightspace.
Texts: Milnor: Morse Theory. — excellent, a little concise.
- Lee: more traditional exposition.

What did you learn last semester? Coursedesc says:
- smooth manifolds? - tangent bundle? - embedding theorems?
- vector fields and differential forms? - intro to Riem metrics?

but I know Jeff doesn't follow that v. closely.

Plan:
- Introduction to Riemannian geometry
- Curvature and geodesics
- Positive and negative curvature
- TBA: Lie groups & symmetric spaces, GGT, metric geometry.

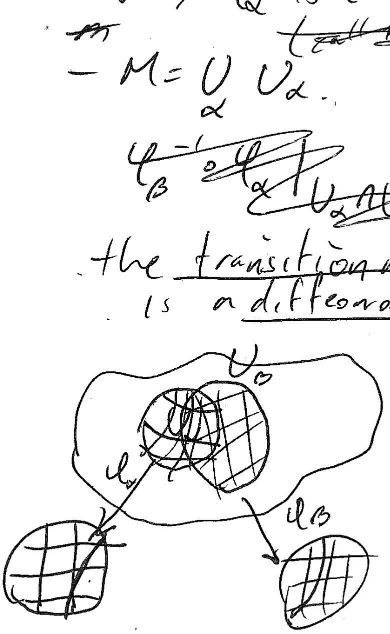
Smooth manifolds: Recall:

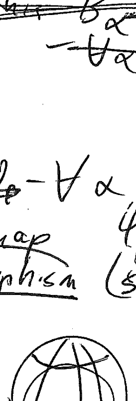
d-Manifold: second countable, Hausdorff, locally Euclidean (locally homeomorphic to a d-ball)

How do we do calculus on a manifold? Need smooth structure.

Def: A smooth d-mfld is a mfld M equipped with a collection of charts $\{(U_\alpha \subset M, \varphi_\alpha : U_\alpha \to \mathbb{R}^d)\}_{\alpha\in A}$ s.t.

- $\forall \alpha$, φ_α is a homeo from U_α to ~~its image~~ an open subset of $\mathbb{R}^d$.
- $M = \bigcup_\alpha U_\alpha$.
- $\forall \alpha, \beta$ s.t. $U_\alpha \cap U_\beta$ is nonempty, the transition map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a diffeomorphism (smooth map w/ smooth inverse).





Charts are local coordinate systems, these have to be compatible.

Standard example: let $f: \mathbb{R}^n \to \mathbb{R}$ a smooth fn. s.t. $\forall x \in \mathbb{R}^n$, if $f(x)=0$, then $\nabla f(x) \neq 0$.

Then: $M = f^{-1}(0)$ is a smooth manifold.
Pf: Implicit Function Theorem (exercise)

<u>Calculus on manifolds</u>: ~~Derivatives of functions~~ Smooth fns: For $U \subset M$ open,

Smooth fns | Def: Let $C^\infty(U)$ = set of smooth functions on U:
~~$\{f: M \to \mathbb{R} \mid f \circ \varphi_\alpha$ is smooth $\forall \alpha\}$~~
where $f: U \to \mathbb{R}$ is smooth iff $f \circ \varphi_\alpha^{-1}$ is smooth $\forall \alpha$

Prop: This is a local condition: f is smooth $\iff$ $\forall x \in U$, $\exists\, \varphi_\alpha$ s.t. $x \in U_\alpha$ and $f \circ \varphi_\alpha^{-1}$ is smooth on a nbhd of $\varphi_\alpha(x)$. (point is transition maps:

~~Likewise $C^\infty(U,N)$ if N is another smooth mfld, can def $C^\infty(U,N)$~~ in terms of charts.

In particular, if $\varphi: U \to \mathbb{R}^n$ is a chart, $\varphi = (x^1, \dots, x^n)$, then $x^i \circ \varphi^{-1}$ = ith coord fn is smooth, so $x^i \in C^\infty(U)$.
~~Closed under~~

Likewise, ~~$C^\infty(U,N)$~~ if (N, ϕ_β) is another smooth mfld, $f: U \subset M \to N$, f is smooth if $\forall \alpha, \beta$ $\phi_\beta \circ f \circ \varphi_\alpha^{-1}$ is smooth. Then $C^\infty(M)$ closed under usual ops.

Vectors, tangent sps | : $\forall x \in M$, let
$$T_xM = \{\gamma: (-\varepsilon, \varepsilon) \to M \mid \gamma(0) = x\}/\sim \quad \gamma \text{ smooth,}$$
where $\gamma_1 \sim \gamma_2 \iff (\varphi_\alpha \circ \gamma_1)'(0) = (\varphi_\alpha \circ \gamma_2)'(0)$ for some α.
$\iff$ " " for all α.
$\iff (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ $\forall f \in C^\infty(M)$.

Given $\gamma: I \to M$, call its class $[\gamma] \in T_{\gamma(0)}M$ $\gamma'(0)$, likewise $\gamma'(t)$.
~~This gives another way to define T_xM which is more convenient for some things~~: That is, $\forall V \in T_xM$, the <u>directional derivative</u> of f in the direction of V is well-defd $\forall f \in C^\infty(M)$, call it Vf.
Further, this acts like a derivative: V is a linear operator $C^\infty(M) \to \mathbb{R}$ which satisfies product rule $V[fg] = Vf \cdot g(x) + f(x) \cdot Vg$.
This gives another way to define T_xM:
Thm: $T_xM = \{D: C^\infty(M) \to \mathbb{R}$ ~~derivs~~ : D is linear,
$$D[f \cdot g] = Df \cdot g(x) + f(x) \cdot Dg\}$$
= set of <u>derivations</u>.

This gives T_xM the structure of a vector space, the <u>tangent space</u> of M at x. Further, ~~a basis of T_xM is given by~~ any chart around x gives a basis: if $\varphi: U \to \mathbb{R}^n$, $\varphi(x) = 0$, ~~then~~ $\varphi = (\varphi^1, \dots, \varphi^n)$ ~~then~~

we define $\partial_i f = \frac{\partial f}{\partial \varphi^i}\Big|_x = \frac{\partial}{\partial x^i}[f\circ\varphi^{-1}]_{x=0}$

- Let TM = tangent bundle of M = collection of all tangent spaces.

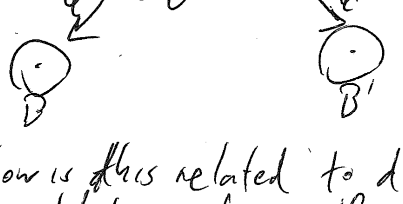
~~Let~~ Prop: TM has a natural smooth structure.

$\boxed{\text{Derivatives of maps}}$ For $f\in C^\infty(M,N)$, we can define the derivative $f_*: TM\to TN$, in several ways.

1) Velocity of curves: For $v\in T_xM$, let $\gamma:(-\varepsilon,\varepsilon)\to M$ s.t. $\gamma(0)=x$, $\gamma'(0)=v$, define $f_*(v)=(f\circ\gamma)'(0)\in T_{f(x)}N$.
(check well-defined)

2) Differential operators: for $h\in C^\infty(N)$, ~~$v\in T_xM$, let~~ $V\in T_xM$, let $f_*(V)[h]=V[h\circ f]$
(check product rule)

3) Coordinate charts: Let $\varphi:U\to B$ be a coord chart near $x\in M$, let $\varphi':U'\to B'$ a coord chart near $f(x)$.
~~Then~~



Then $\varphi'\circ f\circ\varphi^{-1}$ is defined on a nbhd of x, is smooth, so its deriv $D_x[\varphi'\circ f\circ\varphi^{-1}]\in M_{n,n'}(\mathbb{R})$

How is this related to derivatives? Let $\partial_1,\dots,\partial_n\in T_xM$ be std basis from φ, let $\partial_1',\dots,\partial_{n'}'\in T_{f(x)}N$ std basis from φ'. Then any $v\in T_xM$ can be written $v=v^1\partial_1+\dots+v^n\partial_n$.
And
$$f_*(v^1\partial_1+\dots+v^n)=w^1\partial_1'+\dots+w^{n'}\partial_{n'}', \text{ where}$$
$$\begin{pmatrix}w^1\\\vdots\\w^{n'}\end{pmatrix}=M\begin{pmatrix}v^1\\\vdots\\v^n\end{pmatrix}$$
~~This is a little cumbersome.~~

Or if we write M in coords: let m_i^j be coeffs ~~of M~~ s.t. $f_*(\partial_i)=\sum m_i^j\partial_j'$,

~~$f_*(\partial$~~ i.e. $M=\begin{pmatrix}m_1^1 & \cdots & m_n^1\\ \vdots & & \vdots\\ m_1^{n'} & \cdots & m_n^{n'}\end{pmatrix}$.

Then $f_*\left(\sum_i v^i\partial_i\right)=\sum_{ij} v^i m_i^j\partial_j'$, which we abbreviate

$$f_*(v^i\partial_i)=v^i m_i^j\partial_j'$$ — Einstein notation: ~~every~~ if ~~an subscri i, j, k~~ ~~variables~~ an index appears in subscript and superscript, we implicitly sum over that index. Overflow: Questions? Next time: Problem w/ 2nd derivs.

[2025-01-26

Last time: Smooth manifolds and vectors.
Today: Vector fields, derivatives

$\forall x\in M$, $T_xM = \{\gamma:(-\varepsilon,\varepsilon)\to M \mid \gamma(0)=x,\ \gamma \text{ smooth}\}/\sim$

$= \{D: C^\infty(M)\to\mathbb{R} \mid D \text{ linear},\ D[fg] = f(x)D[g] + D[f]g(x)\}$

Let $TM = \coprod_x T_xM$ = tangent bundle of M. ~~Then there~~

Ex: Put a smooth structure on TM. In fact, this is a vector bundle:

~~$\forall x \in M$,~~

There's a natural projection $TM \to M$ ~~A smooth section of π~~

$\pi(T_xM) = x$, $\forall x\in M$, $\pi^{-1}(x)$ is a vect space, local trivializations

~~A vector~~ Let $U\subset M$. A vector field $V\in\mathfrak{X}(U)$ is a smooth section of π, i.e., $\forall x\in U$, $V_x\in T_xM$.

Then V is a ~~an operator~~ differential operator on $C^\infty(U)$, ~~where~~

$f \longmapsto V[f](x) = V_x[f]$

and $V[fg] = fV[g] + V[f]g$

Ex: ~~$V(U)$~~ $\exists$ a bijection from $\mathfrak{X}(U)$ to the set of such ops.

Further, if $\varphi: U\to M$ is a chart, $V\in\mathfrak{X}(U)$, then $\exists\ v^1,\dots,v^d\in C^\infty(U)$ s.t. $V=\sum_i v^i\partial_i$. - any v. field can be locally desc by d coord fns.

Derivatives of maps: Let $f\in C^\infty(M,N)$. We can define the derivative $f_*: TM\to TN$ in a few ways:

1) Curves: For $V\in T_xM$, let $\gamma:(-\varepsilon,\varepsilon)\to M$ s.t. $\gamma(0)=x$, $\gamma'(0)=V$.
Let $f_*(V) = [f\circ\gamma] = (f\circ\gamma)'(0)\in T_{f(x)}N$ (Ex: Check well-defined).

2) Differential operators: for $V\in T_xM$, $\alpha\in C^\infty(N)$ let
$f_*(V): C^\infty(N)\to\mathbb{R}$, $f_*(V)[\alpha] = V[\alpha\circ f]\in T_{f(x)}N$.
(Ex: Check the product rule.)

3) Coordinate charts: Let $\varphi:U\to B$ a chart s.t. $x\in U\subset M^m$,
$\varphi':U'\to B'$ a chart s.t. $f(x)\in U'\subset N^n$.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & U' \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

Then $\varphi'\circ f\circ\varphi^{-1}$ is defined on a nbhd of $\varphi(x)$, is a smooth map. Call this F

Let ~~AB~~ $A = D_{f(x)} f : \mathbb{R}^m \to \mathbb{R}^n$ Then we can write

$$A = \begin{pmatrix} a_1^1 & \cdots & a_m^1 \\ \vdots & & \vdots \\ a_1^n & \cdots & a_m^n \end{pmatrix}$$ — columns are the image of std basis.

~~And (check) is~~ ~~f_*~~ Std basis: Let $\partial_1, \dots, \partial_m \in T_x M$, $\partial_1', \dots, \partial_n' \in T_{f(x)} M$ be std basis. — then

$$\cancel{B}\ f_*(\partial_i) = \sum_j \cancel{\ } a_i^j \partial_j .$$

More generally, $f_*(\sum_i v^i \partial_i) = \sum_i \sum_j v^i a_i^j \partial_j$.

Note: ~~It we~~ All these expressions have prop that if an index appears as sub and super, we're really taking a sum — this is very common, so ~~std notat~~ there's a notation, Einstein summation ~~for~~ where if an index appears twice in a term, we sum over that index — so we write $f_*(v^i \partial_i) = v^i a_i^j \partial_j$.

~~Derivatives of vector fields: Given a curve $\gamma : (-\varepsilon, \varepsilon) \to M$, what's its acceleration at time 0?~~

~~We can't say: Try: Consider $\gamma'(t) = \cancel{\gamma_*(\ )} \in TM$. Then TM is smooth, take $(\gamma')'$? What sort of object do we get?~~

~~$(\gamma')'(0) \in T(TM)$ If $M = \mathbb{R}^n$, there's natural trivialization. then $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, so we can think $\gamma'(t) \in \mathbb{R}^n$, $\gamma''(t) \in \mathbb{R}^n$, etc., but if $M = S^2$, we can't. Instead, there are different notions depending on what you're doing.~~

~~B~~ Def: A diffeomorphism $f : M \to N$ is a smooth map with smooth inverse. Equiv, f is a homeomorphism s.t. $(f_*)_x : T_x M \to T_{f(x)} N$ is ~~also~~ invertible for all x.

If ~~M~~ f is a diffeo from $M \to N$, then M and N are "the same" smooth manifold: — $C^\infty(M) \cong C^\infty(N)$

$TM \cong TN$

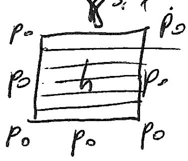
~~etc. if $g : P \to M$, then $f \circ g : P \to N$.~~

every curve, chart, map, etc in M has an analogue in N.

Differential topology studies the equivalence classes of smooth mflds –
– what ~~are~~ quantities (groups, rings, etc.) that are invt under diffeo?

– When is there a map from M to N with certain properties?
– etc.

Key advantage: smooth manifolds are locally nice.
Ex: Thm: $\pi_1(S^2) = 0$
That is, if $\gamma : [0,1] \to S^2$ is a closed curve, with $\gamma(0)=\gamma(1)=p_0$
then ~~there is a homotopy h_t~~ γ is null-homotopic. That is,
$\exists$ a cts family $h_t : [0,1] \to S^2$ $t \in [0,1]$ s.t.
$h_1 = \gamma$, $h_0 = p_0$, $h_t(0) = h_t(1) = p_0$.



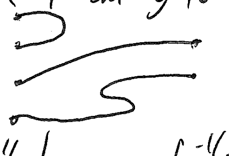
Pf: Let $q \in S^2$ s.t. $q \notin \gamma([0,1])$.
Then $S^2 \setminus q \cong \mathbb{R}^2$. WLOG, suppose $p_0 = 0$.
Then $h_t(s) = t\gamma(s)$ works. //

~~Problem~~ What's the problem with this? – if γ is ~~cts~~, space-filling,
we might have $\gamma([0,1]) = S^2$.
But smooth lets us avoid this:
~~Thm~~ Sard's theorem: Let $f : M \to N$. We say a pt $y \in N$ is
a regular value if $Df_x : T_xM \to T_yN$ is surjective
$\forall x \in f^{-1}(y)$, ~~singular~~ critical value otherwise.
The set of critical values has Lebesgue measure zero.
In particular, if $\dim M < \dim N$, then $f(M)$ consists of
critical values, so it has measure zero.

Prop: Implicit Function Theorem: if ~~x~~ $y \in N$ is a regular value of f, then
$f^{-1}(x)$ is a smooth manifold of dim $\dim M - \dim N$.

App: Let $\dim M = \dim N$, M compact, N compact connected. Let $f : M \to N$, suppose $y \in N$ is regular. ~~Then for $y \in N$, let $\deg_y(f)$~~
Then $f^{-1}(y)$ is a 0-manifold. In fact, $f^{-1}(y)$ is
a finite set of points (Ex: prove).
Let $\deg_y(f) = |f^{-1}(y)| \bmod 2$.

Then $\deg_y(f)$ is invt of y: if $z \in N$ also regular, then
$\exists$ a curve λ from y to z so that $f^{-1}(\lambda)$ is a 1-manifold with
boundary.



– the total # of bdry pts is even,
so $\deg_y(f) + \deg_z(f)$ is even.

$f^{-1}(y)$ $f^{-1}(z)$ Overflow: These are all topological – what about geometric?

~~EDZ~~ [2025-01-27

Last time: Vector bundle, derivatives, diffeomorphisms.

These are all differential topology — to generalize, topology tends to study objects ~~that are locally~~ with no local invariants: every point on ~~the~~ a mfld looks the same wrt topology.
Key question: How do you tell what mfld X is without any local struct?
Today: Diff Geometry — Objects with local structure
~~Key q~~ Easy to tell difference between sphere and cube —
Q: How does the local structure affect the global structure?

~~Top~~ This semester primarily Riemannian Geometry — the extra structure comes from a metric.

Ex: Manifolds in $\mathbb{R}^n$: Let M be a smooth embedded submanifold of $\mathbb{R}^n$. Then M is equipped with a natural metric — if $\gamma: [0,1] \to M$ smooth, we can define $\ell(\gamma) = \int_0^1 \|\gamma'(t)\| dt$. ~~Intuitively this should let us distinguish~~ points on M — how? How do we do geom on M?

~~Geodesics on M / short~~ ~~One way:~~ For example: Shortest paths on M:

Q: Given $p, q \in M$, minimize $\ell(\gamma)$ over curves with $\gamma(0) = p$, $\gamma(1) = q$.

First, ~~not~~ note that ℓ is tricky to minimize because there isn't a unique minimum — if γ minimizes $\ell(\gamma)$, then so does any parametrization of γ.

We thus define the Energy of $\gamma: [a,b] \to M$ by
$$E(\gamma) = E_a^b(\gamma) = \frac{1}{2}\int_a^b \|\gamma'(t)\|^2 dt$$
By Cauchy-Schwarz, if $\gamma: [a,b] \to M$, then
$$\int_a^b \|\gamma'(t)\| dt \leq \sqrt{\int_a^b \|\gamma'(t)\|^2 dt} \cdot \sqrt{\int_a^b 1\, dt}$$
$$= \sqrt{2E_a^b(\gamma)} \cdot \sqrt{b-a},$$
~~with~~ That is, ~~equality iff $\|\gamma'(t)\|$ is constant~~, so
$$E(\gamma) \geq \frac{\ell(\gamma)^2}{2(b-a)},$$
with equality iff $\|\gamma'(t)\|$ is constant.

~~If γ~~ — so ~~the γ that minimi~~ a minimizer of energy ~~is a length minimizing curve parametrized with constant speed.~~
So for any curve, the energy-min param has constant speed
$\Rightarrow$ an energy minimizer is a const speed length minimizer.

So we minimize E. If M is closed, connected, then Arzela-Arcoli $\Rightarrow \exists$ an ~~extende~~ energy minimizer $\gamma : [0,1] \to M$. What can we say about γ?

Calculus of Variations: if γ minimizes E, then any perturbation of γ increases energy. So we want to understand perturbations.

Let $h_u : [0,1] \to \mathbb{R}^n$, $u \in (-\varepsilon, \varepsilon)$ be a family of curves ~~s.t. $h_u(0) = p$, $h_u(1) = q$ $\forall u$ and $h_u(t)$ depends~~ s.t. $H(u,t) = h_u(t)$ is smooth. ~~and $h_0 = \gamma$.~~

~~(a variation of γ.)~~

Then $E(h_u) \geq E(h_0)$ $\forall u$ — we calculate:

$$E(h_u) = \frac{1}{2}\int_0^1 \|h_u'(t)\|^2 dt = \frac{1}{2}\int_0^1 \left\| \frac{\partial H}{\partial t} \right\|^2 dt$$

$$\frac{d}{du}E(h_u) = \frac{1}{2}\int_0^1 \frac{\partial}{\partial u}\left\langle \frac{\partial H}{\partial t} \middle| \frac{\partial H}{\partial t} \right\rangle dt = \int_0^1 \left\langle \frac{\partial^2 H}{\partial u \partial t} \middle| \frac{\partial H}{\partial t} \right\rangle dt$$

This is a product - we can use IBP: $\frac{\partial}{\partial t}\left\langle \frac{\partial H}{\partial u} \middle| \frac{\partial H}{\partial t} \right\rangle = \left\langle \frac{\partial^2 H}{\partial u \partial t} \middle| \frac{\partial H}{\partial t} \right\rangle + \left\langle \frac{\partial H}{\partial u} \middle| \frac{\partial^2 H}{\partial t^2} \right\rangle$

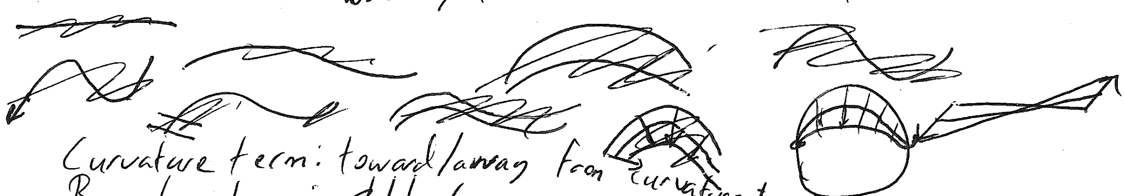
$$\Rightarrow \frac{d}{du}E(h_u) = \int_0^1 \frac{\partial}{\partial t}\left\langle \frac{\partial H}{\partial u} \middle| \frac{\partial H}{\partial t} \right\rangle dt - \int_0^1 \left\langle \frac{\partial H}{\partial u} \middle| \frac{\partial^2 H}{\partial t^2} \right\rangle dt$$

$$= \left\langle \frac{\partial H}{\partial u} \middle| \frac{\partial H}{\partial t} \right\rangle \Big|_{t=0}^{1} - \int_0^1 \left\langle \frac{\partial H}{\partial u} \middle| \frac{\partial^2 H}{\partial t^2} \right\rangle dt$$

~~Since $E(h_0)$ is a minimum, $\frac{\partial}{\partial u}[E(h_u)]_{u=0} = 0$. Plug in 0:~~

And we can name these fields: $\frac{\partial H}{\partial t} = V$ = velocity, $\frac{\partial^2 H}{\partial t^2} = A$ = acceleration, $\frac{\partial H}{\partial u} = U$ = variation field:

$$= \underbrace{\langle V(1) | U(1) \rangle - \langle V(0) | U(0) \rangle}_{\text{boundary term}} - \underbrace{\int_0^1 \langle U | A \rangle dt}_{\text{curvature term}}$$



Curvature term: toward/away from curvature direction

Boundary term: ~~which wa~~ endpoints move in/out.

~~Specifically:~~ In this situation, suppose $\gamma : [0,1] \to M$ is an energy minimizer, $h_u : [0,1] \to M$, $u \in (-\varepsilon, \varepsilon)$ is a perturb s.t. $h_u(0) = p$, $h_u(1) = q$ $\forall u$, $h_0 = \gamma$. Then $E(h_u) \geq E(h_0)$ $\forall u$.

$\Rightarrow \frac{d}{du}\Big|_{u=0} E(h_u) = 0 \quad \forall$ such h_u

$\Rightarrow$ ~~$0 = \int_0^1 \langle \ldots \gamma''$~~ $0 = -\int_0^1 \langle \frac{\partial H}{\partial u} | \gamma''(t) \rangle \, dt$

$\forall$ such h_u

But we have a lot of freedom to construct h_u — as long as $\frac{\partial H}{\partial u}$ is tangent to M.

~~So~~ If this int is 0 for all h_u, then $\gamma''(t) \perp TM \quad \forall t$.

~~= this is the geodesic equation~~

~~Conversely,~~ If $\pi_{TM} : \mathbb{R}^n \to TM$ is orthogonal projection we can write $\pi_{TM}(\gamma''(t)) = 0 \quad \forall t$.

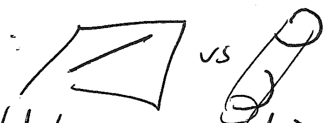
Def: if $\pi_{TM}(\gamma''(t)) = 0 \quad \forall t$, then γ is a geodesic.

(possibly explain lack of ~~gtt in~~ diff top)?

Let me point out something interesting here.

Def: Let $M, N \subset \mathbb{R}^n$. A map $\varphi : M \to N$ is an isometry if φ is a homeo and $\ell(\varphi \circ \gamma) = \ell(\gamma) \quad \forall$ curves γ.

Then: γ'' isn't preserved by isometry:



But all of the calculus we did should be preserved:

if h_u is a variation of γ with fixed end pts, $\bar{h}_u = \varphi \circ h_u$, $\bar{\gamma} = \varphi \circ \gamma$.

then $\frac{d}{du} E(\bar{h}_u) = \frac{d}{du} E(h_u)$

$$-\int_0^1 \langle \frac{\partial \bar{h}}{\partial u} | \bar{\gamma}''(t) \rangle dt = -\int_0^1 \langle \frac{\partial h}{\partial u} | \gamma''(t) \rangle dt$$

$$\int_0^1 \langle \varphi_*\left(\frac{\partial h}{\partial u}\right) | \bar{\gamma}''(t) \rangle dt = \int_0^1 \langle \frac{\partial h}{\partial u} | \gamma''(t) \rangle dt$$

In fact, $\forall t$, $\forall v \in T_{\gamma(t)} M$,

$$\langle \varphi_*(v) | \bar{\gamma}''(t) \rangle = \langle v | \gamma''(t) \rangle$$

$$\Rightarrow \pi_{TN}(\bar{\gamma}''(t)) = \varphi_*(\pi_{TM}(\gamma''(t))) \quad \forall t.$$

So $\pi(\gamma''(t))$ is preserved — the tangential component of acceleration: — there is a way to calculate second derivs based on lengths

~~Next:~~

Reverse this next time: define tangential second derivative, show invariance

(2025-02-03

Last week: Derivatives:

~~Pr~~ If M, N smooth mflds, $\varphi: M \to N$, $\gamma: I \to M$ a curve, then
$$(\varphi \circ \gamma)' = \varphi_*(\gamma')$$
(topology) This doesn't work for second deriv – we can't define γ''

But:

Prop: If $M, N \subset \mathbb{R}^n$, $\varphi: M \to N$ is an isometry, $\gamma: I \to M$, then
$$\pi_{TN}((\varphi \circ \gamma)'') = \varphi_*(\pi_{TM}(\gamma'')).$$

This week: generalize.

Def: Let M be a smooth manifold. A Riemannian metric on M assigns a pos. def inner product to ~~M~~ each space T_pM in a smooth way. $\forall x \in M$, there is a $g_x: T_xM \times T_xM \to \mathbb{R}$ s.t. $\forall u, v, w \in T_xM$, $\forall a, b \in \mathbb{R}$,

1 – $g_x(u,v) = g_x(v,u)$ (symmetric)
2 – $g_x(au+bv, w) = a g_x(u,w) + b g_x(v,w)$ (bilinear)
3 – $g_x(u,u) \geq 0$ and $g_x(u,u) = 0 \Rightarrow u = 0$ (positive-definite)
4 – $\forall U, V \in \mathfrak{X}(M)$, $x \mapsto g_x(U_x, V_x)$ is smooth.

We write $g_x(u,v)$ as $\langle u|v\rangle$, define $\|v\| = \sqrt{\langle v|v\rangle}$
$$\cos\angle(u,v) = \frac{\langle u|v\rangle}{\|u\|\,\|v\|}.$$
$$\ell(\gamma) = \int \|\gamma'(t)\|\,dt.$$

~~Ex~~ Inner product facts:

– Given $V_1, \dots, V_k \in \mathfrak{X}(U)$ s.t. $(V_1)_x, \dots, (V_k)_x$ are linearly independent we can apply Gram-Schmidt to construct orthonormal fields $W_1, \dots, W_k$.

(in particular, $\forall x, y \in M$, ~~there is an isometry~~ $T_xM \cong T_yM$.)

– $\langle\cdot|\cdot\rangle$ is determined by $\|\cdot\|$:
$$\langle v|w\rangle = (\|v+w\|^2 - \|v-w\|^2)/4.$$

n^2 smooth fns

~~so you'll see~~

– ~~$\langle\cdot|\cdot\rangle$ is determined by $\langle e_i|e_j\rangle$ for any basis.~~

– ~~$\langle\cdot|\cdot\rangle$ is determined by $g_{ij} = \langle\partial_i|\partial_j\rangle \in C^\infty(U)$~~

– Let $\varphi: U \to \mathbb{R}^n$ be a chart, $\partial_1, \dots, \partial_n$ std basis. Then $\langle\cdot|\cdot\rangle$ is determined by the n^2 smooth fns
$$g_{ij} = \langle\partial_i|\partial_j\rangle, \text{ i.e. } \langle v^i\partial_i|w^j\partial_j\rangle = v^i w^j g_{ij}$$

So we'll frequently write metrics in terms of the quadratic form $\|\cdot\|^2$: ~~dg~~

$$dg^2 = dx^2 + dy^2 \quad \cdot \quad \|v\|^2 = dx(v) + dy(v) = v_x^2 + v_y^2$$

$v_x^2 + v_y^2$ ~~$v = (v_x, v_y)$~~ ~~$\|(x,y)\|$~~

Any metric on $\mathbb{R}^2$ can be written
$$dg^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2$$
for some $E, F, G \in C^\infty(\mathbb{R}^2)$, $\det\begin{vmatrix} E & F \\ F & G \end{vmatrix} > 0$.

~~Def: For $\gamma: I \to M$, let $\ell(\gamma) = \int_I \|\gamma'(t)\|\,dt$.~~

Claim: A Riemannian metric is enough structure to define a canonical second derivative.
- what does this mean?

<u>Connections</u>: A <u>connection</u> on M is a collection of functions $\nabla: T_pM \times \mathcal{V}(M) \to T_pM$, $p \in M$, written $\nabla_{X_p} Y$ for $X_p \in T_pM$, $Y \in \mathcal{V}(M)$, s.t.

1. ∇ is bilinear in X_p and Y.
2. (smoothness) if $X, Y \in \mathcal{V}(M)$, then $\nabla_X Y = (p \mapsto \nabla_{X_p} Y) \in \mathcal{V}(M)$
3. (product rule) $\nabla_{X_p} fY = (X_p f) Y + f \nabla_{X_p} Y$.

i.e., a way to take directional derivatives of vector fields.

Ex: If $M = \mathbb{R}^n$, define the <u>trivial connection</u>

$$\nabla_{X_p} Y = \frac{d}{dt}\Big|_{t=0} Y_{p+tX_p}$$ (identifying $T_p\mathbb{R}^n \cong \mathbb{R}^n$, viewing $Y \in \mathcal{V}(\mathbb{R}^n)$ as a fn $Y: \mathbb{R}^n \to \mathbb{R}^n$)

Ex: If $M \subset \mathbb{R}^n$ is a smooth submanifold, define <u>tangential connection</u> as follows:

For $Y \in \mathcal{V}(M)$ let $\overline{Y} \in \mathcal{V}(U)$ be an extension of Y to a nbhd $M \subset U \subset \mathbb{R}^n$. Define
$$\nabla^T_{X_p} Y = \pi_{T_pM}(\nabla_{X_p} \overline{Y})$$
(need to project because $\nabla_{X_p} \overline{Y}$ need not be in T_pM)

Claim: this is well-defined, bilinear, product rules.

But there are a lot of connections:
Let $\varphi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ a chart, $\partial_1, \dots, \partial_n \in \mathfrak{V}(U)$ std basis. Let $\Gamma^k_{ij} \in C^\infty(U)$ arbitrary smooth functions.

Then $\exists$ a connection ∇ s.t. $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$.

Namely, if $X = f^i \partial_i$, $Y = g^j \partial_j$. Then product + bilinear imply:

$$\nabla_X Y = \nabla_{f^i\partial_i} [g^j \partial_j] = f^i \nabla_{\partial_i} [g^j \partial_j]$$
$$= f^i \left[\partial_i g^j \cdot \partial_j + g^j \nabla_{\partial_i} \partial_j \right]$$
$$= f^i \partial_i g^j \cdot \partial_j + f^i g^j \Gamma^k_{ij} \partial_k$$

(Check that this satisfies product, bilinear)
So a connection is locally described by n^3 smooth functions.

But if we have one, we can define ~~derivs along~~ derivs along curves:
__Covariant derivative__: For $\gamma : I \to M$, let $\mathfrak{V}(\gamma)$ = {smooth v. fields along γ}, i.e.

$$\mathfrak{V}(\gamma) = \{ V : I \to TM \mid V(t) \in T_{\gamma(t)} M,\ V \text{ is smooth} \}$$

(i.e. $t \mapsto V(t) f$ is smooth for all $f \in C^\infty(M)$ or $\forall t_0$, $\exists$ a chart $U \ni \gamma(t_0)$, smooth fns $v^1, \dots, v^n$ s.t. $V(t) = v^i \partial_i$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.)

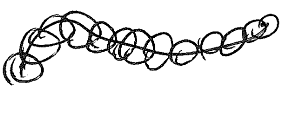
Then: Lemma: Given a connection ∇, curve $\gamma : [0,1] \to M$. $\exists!$ __covariant derivative__ $D_t : \mathfrak{V}(\gamma) \to \mathfrak{V}(\gamma)$ s.t.

1. D_t is linear
2. $D_t [fV] = \frac{df}{dt} V + f D_t V \quad \forall f \in C^\infty(M),\ V \in \mathfrak{V}(\gamma)$.
3. If V is the restriction of $\bar{V} \in \mathfrak{V}(M)$, then
$$D_t V = \nabla_{\gamma'(t)} \bar{V}.$$

Pf: ~~In any patch,~~ ③ implies $D_t \partial_i = \nabla_{\gamma'(t)} \partial_i$. ~~Pick a patch~~ Consider patch $\varphi : U \to \mathbb{R}^n$. ~~Let $V \in \mathfrak{V}(\gamma)$.~~ Suppose $\gamma : I \to U$, then $V = v^i \partial_i$. ~~Then.~~ ~~In that patch,~~ Pick a patch with $V = v^i \partial_i$. Calculate:

$$D_t [v^i \partial_i] \underset{①,②}{=} \frac{dv^i}{dt} \partial_i + v^i \cdot D_t \partial_i \underset{③}{=} \frac{dv^i}{dt} \partial_i + v^i \nabla_{\gamma'(t)} \partial_i$$

So D_t exists, is unique for curves in U. For arb curve:



cover by patches, define on each patch — then well-defined by uniqueness.

2025-02-05

Last time: Connections:
$\nabla_X Y$ = derivative of Y in direction of X.
- many possible ∇'s.

~~Today: Given ∇, what can we do?~~
~~Today: Given ∇, what can we do with it?~~
But: Fundamental Lemma of Riemannian Geometry:
Let (M,g) be a ~~smooth~~ Riemannian manifold. Then $\exists!$ connection ∇ on M which is torsion-free, and compatible with the metric. When $M \subset \mathbb{R}^n$, $\nabla = \nabla^T$.

We need to explain torsion-free, compatible — ~~we'll do that today~~ ~~we'll start with compatible~~ but before that we need to talk more ~~about connections~~ about what we can do with connections:

~~Applications of connections~~:
① Derivatives along curves:
For $\gamma: I \to M$, let $\mathcal{V}(\gamma) = \{$smooth v. fields on $\gamma\}$,
i.e., ~~smooth maps~~ $= \{V: I \to TM$ ~~s.t.~~ $\mid V(t) \in T_{\gamma(t)}M \ \forall t,\ V$ is smooth$\}$ ~~$\forall (t \mapsto V(t) f)$ is smooth $\forall f \in C^\infty(M)\}$~~

Any such V can locally be written in coords:
~~Then V can locally be written in coordinates:~~
Then $\forall t_0 \in I$, $\exists$ a chart $\varphi: U \to \mathbb{R}^n$, $\varepsilon > 0$ and fns $v^1, \dots, v^n$ s.t. $\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, $V(t) = v^i \partial_i$
$= v^i(t) (\partial_i)_{\gamma(t)}$

(conversely, if $V: I \to TM$ can be written this way, then $V \in \mathcal{V}(\gamma)$).

Lemma: Given a connection ∇ and a curve $\gamma: I \to M$, there is a unique covariant derivative $D_t: \mathcal{V}(\gamma) \to \mathcal{V}(\gamma)$ s.t.
① D_t is linear
② $\forall f \in C^\infty(I)$, $D_t[fV] = \frac{df}{dt} V + f D_t V$
③ if V is the restriction of some $\tilde{V} \in \mathcal{V}(M)$, then $D_t V = \nabla_{\gamma'(t)} \tilde{V}$.

Pf: Let $\varphi: U \to \mathbb{R}^n$ be a patch, and suppose that $\gamma \subset U$.
Then for $V \in \mathcal{V}(\gamma)$, $\exists v^1, \dots, v^n$ s.t. $V = v^i \partial_i$, so

$$D_t V = D_t[v^i \partial_i] = \frac{dv^i}{dt} \partial_i + v^i D_t \partial_i = \frac{dv^i}{dt} \partial_i + v^i \nabla_{\gamma'(t)} \partial_i.$$

That is, lemma holds on any patch. If γ arbitrary,

then we can ~~cover~~ γ by patches — on each patch we have a unique D_t. By uniqueness, the D_t's agree on overlaps, so we can define D_t on all of γ. //

② <u>Parallel fields</u> Key fact about connections: we can't generally construct parallel vector fields on a whole manifold — we can construct parallel fields on a path.

$V \in \mathcal{V}(\gamma)$ is <u>parallel</u> along γ if $D_t V = 0$.

Lemma: Let γ be a smooth curve. Then $\forall t_0, c \in T_{\gamma(t_0)}M$, $\exists ! \, V \in \mathcal{V}(\gamma)$ s.t. $D_t V = 0$ and $V(t_0) = c$.

Pf: Again, first prove for a patch, then for the whole: Let $\varphi : U \to \mathbb{R}^n$ suppose $\gamma \subset U$. ~~Then any V can be written~~ Let $V = v^i \partial_i$. write $\gamma = (\gamma^1, \dots, \gamma^n)$

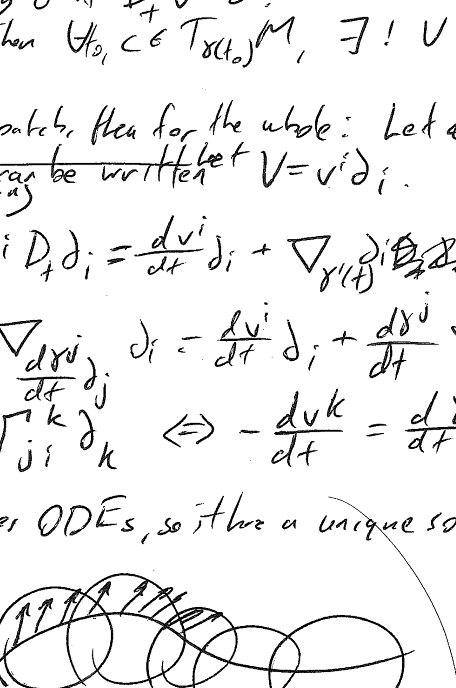
Then

$$V \text{ is parallel} \Leftrightarrow 0 = D_t V = \frac{dv^i}{dt}\partial_i + v^i D_t \partial_i = \frac{dv^i}{dt}\partial_i + \nabla_{\gamma'(t)} \partial_i$$

$$= \frac{dv^i}{dt}\partial_i + \nabla_{\frac{d\gamma^j}{dt}\partial_j} \partial_i = \frac{dv^i}{dt}\partial_i + \frac{d\gamma^j}{dt}\nabla_{\partial_j}\partial_i = 0$$

$$\Leftrightarrow -\frac{dv^k}{dt}\partial_k = \frac{d\gamma^j}{dt}\Gamma_{ji}^{k}\partial_k \Leftrightarrow -\frac{dv^k}{dt} = \frac{d\gamma^j}{dt}\Gamma_{ji}^{k} \quad \forall k.$$

This is a system of 1st order ODEs, so there is a unique solution for any initial condition



In general, cover by patches, construct on each patch. //

~~In particular, if $W_1, \dots, W_n \in \mathcal{V}(\gamma)$ are parallel fields that span~~

~~Convenient Fact: ③ If $E_1, \dots, E_n \in \mathcal{V}(\gamma)$ are parallel fields and~~

~~$V = v^i E_i$, then $D_t V = \frac{dv^i}{dt}E_i + v^i D_t E_i$~~

③ <u>Parallel transport</u>: Let $\gamma : I \to M$, $t_0, t_1 \in I$. For $c \in T_{\gamma(t_0)}M$, let V_c be parallel field with $V_c(t_0) = c$.

Let $P_{t_0,t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$, $P_{t_0,t_1}(c) = V_c(t_1)$. Then

① $\forall t_0, t_1, t_2 \in I$, $P_{t_2,t_1} \circ P_{t_1,t_0} = P_{t_2,t_0}$. ② P_{t_0,t_1} is a linear isom.

③ $D_t V(t_0) = \left.\frac{d}{dt}\right|_{t=t_0} P_{t_0,t}(V(t))$

Pf: ① Exercise.

② By ①, $P_{t_0,t_1} \circ P_{t_1,t_0} = id$. ~~State~~ $\cancel{\nabla_{a+b}} = \cancel{\nabla_a + \nabla_b}$,

~~Check that ∇_a~~ Ex: $V_{a+b} = V_a + V_b$..

③ Based on parallel frames: let $e_1, \cdots, e_n \in T_{\gamma(t_0)}M$ be a basis.
Let $E_i = V_{e_i}$ – ~~state~~ by ②, $E_1(t), \cdots, E_n(t)$ is a basis $\forall t$,
So if $W \in \mathcal{V}(\gamma)$, $\exists w^i$ s.t. $W = w^i E_i$.

On one hand, $D_t W(t_0) = \frac{dw^i}{dt}(t_0) E_i(t_0) + \cancel{w^i D_t E_i(t_0)}$,

OTOH: $P_{t_0,t}(W(t)) = P_{t_0,t}(w^i E_i(t)) = w^i(t) P_{t_0,t}(E_i(t))$
$= w^i(t) E_i(t_0)$

$\frac{d}{dt}\Big|_{t=t_0} P_{t_0,t}(W(t)) = \frac{dw^i}{dt}(t_0) E_i(t_0) = D_t W(t_0)$. //

~~So Now we can define compat~~

~~Def: A connection ∇ on M is compatible with a metric g if:~~
~~① any of + We say that ∇ is compatible with g if any of the~~

We say that ∇ is compat with g if P_{t_0,t_1} is an isometry $\forall t_0, t_1$.
(i.e. $\langle P_{t_0,t_1}(v) | P_{t_0,t_1}(w)\rangle = \langle v|w\rangle \ \forall t_0, t_1, v, w$.)

Lemma: TFAE:

① ∇ is compatible with g.

② $\forall \gamma$, if $V_1, V_2 \in \mathcal{V}(\gamma)$ are parallel, then $\langle V_1 | V_2\rangle$ is const.

③ $\forall V \in \mathcal{V}(\gamma)$ V is parallel then $\|V\|$ is const.

④ $\forall V, W \in \mathcal{V}(\gamma)$, $\frac{d}{dt}\langle V|W\rangle = \langle D_t V | W\rangle + \langle V | D_t W\rangle$

⑤ $\forall X, Y, Z \in \mathcal{V}(M)$, $X[\langle Y|Z\rangle] = \langle \nabla_X Y | Z\rangle + \langle Y | \nabla_X Z\rangle$.

Pf: ①⇔②⇔③ Exercise, ④⇔⑤ exercise.

④⇒②: $\frac{d}{dt}\langle V_1 | V_2\rangle = \langle D_t V_1 | V_2\rangle + \langle V_1 | D_t V_2\rangle$,
so if V_1, V_2 parallel, this is zero.

②⇒④ is interesting:

If ② holds, and $e_1, \cdots, e_n \in T_{\gamma(t_0)}M$ is an orthonormal basis,
then we can extend ~~the~~ them to $E_1, \cdots, E_n \in \mathcal{V}(\gamma)$
a parallel orthonormal frame on γ: – $E_1(t), \cdots, E_n(t)$ are orth. basis of $T_{\gamma(t)}M$ $\forall t$.

Let ~~be~~ $V = v^i E_i$, $W = w^j E_j$. Then $D_t V = \frac{dv^i}{dt} E_i + \cancel{v^i D_t E_i}$

$\frac{d}{dt}\langle V|W\rangle = \frac{d}{dt} \sum v^i w^i = \sum_i \frac{dv^i}{dt} w^i + v^i \frac{dw^i}{dt}$ $\quad D_t W = \frac{dw^j}{dt} E_j$

$= \langle D_t V | W\rangle + \langle V | D_t W\rangle = \cancel{\sum_i \frac{dv^i}{dt}}$ //

2025-02-10

Last time: Fundamental Lemma of Riemannian Geometry:
Let (M,g) be a Riemannian manifold. Then $\exists!$ connection ∇ which is torsion-free and compatible with g.

- Compatible with g: parallel transport preserves lengths/angles.
- Torsion-free: explain today - based on Lie bracket:

Lie bracket: Another way to differentiate vector fields:
For $X, Y \in \mathcal{V}(M)$, let $[X,Y] = XY - YX$, i.e. $[X,Y]$ is the derivation s.t. $[X,Y]f = X[Y[f]] - Y[X[f]]$.
Check: - This is a derivation,
- This is antisymmetric: $[X,Y] = -[Y,X]$.
- Bilinear over $\mathbb{R}$: $[aX+bY, Z] = a[X,Z] + b[Y,Z]$.
- Satisfies a product rule:
let $\mathcal{L}_X(Y) = [X,Y]$. Then $\mathcal{L}_X(fY) = (Xf)Y + f\mathcal{L}_X(Y)$

$$[X, fY] = Xf\cdot Y + f[X,Y].$$

~~Geometrically~~
~~Ex: Let~~ - Let $\varphi: U \to \mathbb{R}^n$ be a chart, $\partial_1, \dots, \partial_n \in \mathcal{V}(U)$.
Then ~~$[\partial_i, \partial_j] = 0 \ \forall i,j$.~~

$$[\partial_i, \partial_j]f = \partial_i\partial_j f - \partial_j\partial_i f = \frac{\partial^2}{\partial x^i \partial x^j} f - \frac{\partial^2}{\partial x^j \partial x^i} f = 0$$

So: $\mathcal{L}_{\partial_i}(w^j\partial_j) = (\partial_i v^j)\partial_j + v^j[\partial_i, \partial_j] = (\partial_i v^j)\partial_j$.

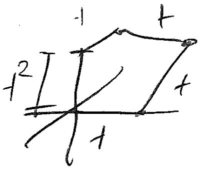
- Lie deriv along a coord field ∂_i is ∂_i applied coordinatewise.
(If you choose a chart s.t. $V = \partial_1$, then $\mathcal{L}_V = \mathcal{L}_{\partial_1}$.)
~~Ex: $X = \partial_x$, $Y = \partial_y + x\partial_z$, $Z = \partial_z$.~~
Likewise: ~~Ex: Heisenberg?~~
$$[v^i\partial_i, w^j\partial_j] = (v^i\partial_i w^j)\partial_j - (w^j\partial_j v^i)\partial_i$$

~~Two~~ Standard interpretations based on flows:
Def: If $V \in \mathcal{V}(U)$, then $\forall x \in U$, $\exists \varepsilon > 0$, $\gamma_x: (-\varepsilon, \varepsilon) \to U$ s.t. γ_x is a flow line of V:
$\gamma_x(0) = x$, $\gamma_x'(t) = V_{\gamma(t)}$. Let $\Phi_V^t(x) = \gamma_x(t)$, then $\forall u \in U$
$(x,t) \mapsto \Phi_V^t(x)$ is smooth on a nbhd of $(u, 0)$.
(If M compact, $V \in \mathcal{V}(M)$, then $\Phi_V^t(x)$ is smooth on $M \times \mathbb{R}$)

Thm: Prop: When t is small,
$$\Phi^t_{-Y}\circ\Phi^t_{-X}\circ\Phi^t_{Y}\circ\Phi^t_{X} \approx \Phi^{t^2}_{[X,Y]}.$$

Ex: $M=\mathbb{R}^3$, $\partial_x, \partial_y, \partial_z \in \mathcal{V}(\mathbb{R}^3)$ $\quad X=\partial_x$, $Y=\partial_y + x\partial_z$, $Z=\partial_z$.

$[X,Y]=Z$



~~Ex: If Φ_X and Φ_Y commute: $\Phi^s_X\circ\Phi^t_Y=\Phi^t_Y\circ\Phi^s_X$, then there's a map $\alpha:(s,t)\mapsto\Phi^s_X\circ\Phi^t_Y(u)$~~

In particular, if $[X,Y]=0$, there's a remarkable thm of Frobenius:

Frobenius Lemma: If $[X_i,X_j]=0$ $\forall i,j$ on U, and $u\in U$, then ~~∃ map~~

- For suff. small s,t, ~~v suff close to u~~,
$$\Phi^s_{X_i}\circ\Phi^t_{X_j}=\Phi^t_{X_j}\circ\Phi^s_{X_i} \text{ on a nbhd of } u.$$

- So $\exists$ a map
$$f(t^1,\dots,t^k)=\Phi^{t^1}_{X_1}\circ\cdots\Phi^{t^k}_{X_k}(u) \text{ s.t. } \frac{\partial f}{\partial t^i}=X_i \ \forall i.$$

- In particular, if $X_1(p),\dots,X_n(p)$ are a basis of T_pM and $[X_i,X_j]=0$, then $\exists$ a chart near p s.t. $\partial_i=X_i$ $\forall i$.

i.e., $[X,Y]$ measures when we can extend X,Y to a chart.

Def: The torsion of ∇ is ~~given by~~ 2-tensor s.t. $\forall X,Y\in\mathcal{V}(M)$.
$$\tau(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y]$$

This is *a priori* a function $\tau:\mathcal{V}(M)\times\mathcal{V}(M)\to\mathcal{V}(M)$, but in fact it's a ~~funct~~ tensor — i.e., $\tau(X,Y)_p$ is a bilinear fn of X_p, Y_p.

Pf: Check that τ is alternating, bilinear ~~of~~ over $\mathbb{R}$.
Let $f\in C^\infty(U)$ — claim that $\tau(X,fY)=f\tau(X,Y)$.
$$\tau(X,fY)=\nabla_X fY-\nabla_{fY}X-[X,fY]$$
$$=\cancel{Xf\,Y}+f\nabla_X Y-f\nabla_Y X-\cancel{Xf\,Y}-f[X,Y]$$
$$=f\tau(X,Y)$$

So if $X=f^i\partial_i$, $Y=g^j\partial_j$, then $\tau(X,Y)=f^ig^j\tau(\partial_i,\partial_j)$
$$\tau(X,Y)_p=f^i(p)g^j(p)\tau(\partial_i,\partial_j)$$ depends only on ~~$f^i(p), g^j(p)$~~ X_p, Y_p.

So τ assigns a bilinear form $\tau_p : T_pM \times T_pM \to T_pM$ to each $p \in M$.

Other advantage: ~~easy~~ we can calculate τ by choosing easy v.f.s.
~~so~~ ~~$\exists$~~ If $\varphi: U \to \mathbb{R}^n$ a chart, $\partial_1, \dots, \partial_n$ std basis.
$$\tau(\partial_i, \partial_j) = \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i - \cancel{[\partial_i, \partial_j]}$$

~~Torsion measures~~ So ~~$\nexists$~~ ∇ torsion-free $\iff \nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i \ \forall i,j$
(Exercises: locally parallel stuff)

Fundamental lemma: $\exists!$ torsion-free compatible connection.
Pf: Let $\varphi: U \to \mathbb{R}^n$ a coord chart. Suppose ∇ is torsion free, compatible on U.
Then $\nabla_{\partial_j}\partial_i = \nabla_{\partial_i}\partial_j$. (torsion-free)

Compatible: $\partial_i\langle\partial_j|\partial_k\rangle = \langle\nabla_{\partial_i}\partial_j|\partial_k\rangle + \langle\partial_j|\nabla_{\partial_i}\partial_k\rangle$
Permute: $\partial_j\langle\partial_k|\partial_i\rangle = \langle\nabla_{\partial_j}\partial_k|\partial_i\rangle + \langle\partial_k|\nabla_{\partial_j}\partial_i\rangle$
$\qquad\partial_k\langle\partial_i|\partial_j\rangle = \langle\nabla_{\partial_k}\partial_i|\partial_j\rangle + \langle\partial_i|\nabla_{\partial_k}\partial_j\rangle$
Note: common terms. So
$$\partial_i\langle\partial_j|\partial_k\rangle + \partial_j\langle\partial_k|\partial_i\rangle - \partial_k\langle\partial_i|\partial_j\rangle = 2\langle\nabla_{\partial_i}\partial_j|\partial_k\rangle.$$

That is, $\langle\nabla_{\partial_i}\partial_j|\partial_k\rangle$ is determined by $g\ \forall i,j,k \Rightarrow$
$\nabla_{\partial_i}\partial_j$ is determined by g. $\Rightarrow \nabla$ is unique.

Existence: Enough to define ~~the~~ Γ^k_{ij} s.t. $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$.

Let $g_{ij} = \langle\partial_i|\partial_j\rangle$, let $(g^{ij}) = (g_{ij})^{-1}$ (matrix inverse) — then $g_{ij}g^{jk} = \delta^k_i$ (identity matrix)
Rewrite:
$$\tfrac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}) = \langle\nabla_{\partial_i}\partial_j|\partial_k\rangle = \langle\Gamma^l_{ij}\partial_l|\partial_k\rangle = \Gamma^l_{ij}g_{kl}.$$
Multiply by g^{mk} and sum:
$$\tfrac{1}{2}g^{mk}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}) = \Gamma^l_{ij}g_{kl}g^{mk} = \Gamma^l_{ij}\delta^m_l = \Gamma^m_{ij}$$
— so for any metric on U, $\exists!$ connection.

Finally, cover M by charts, construct ∇ on each. By uniqueness, these ~~are all~~ agree on overlaps. //
— this is the Levi-Civita connection

(2025-02-12

Last time: Levi-Civita connection: FLRG: $\exists!$ torsion-free, compat connection. Let $\varphi: U \to \mathbb{R}^n$ a chart, $g_{ij} = \langle \partial_i | \partial_j \rangle$. Then

Ex: When $M \subset \mathbb{R}^n$, this is tangential connection.

$$\langle \nabla_{\partial_i} \partial_j | \partial_k \rangle = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$$

Further, ∇ is unique on any patch - so if ~~$U \subset M$, $V \subset N$, $\varphi: U \to V$ is an isometric~~ $\varphi: M \to N$ is an isometry, or $U \subset M$, $V \subset N$, $\varphi: U \to V$ an isometry, then

$$\nabla_{\varphi_*(X)} \varphi_*(Y) = \varphi_*(\nabla_X Y) \quad \forall X, Y \in \mathcal{V}(U).$$

Q: What does ∇ tell us about the geometry of M?

Geodesics: We say $\gamma: I \to M$ is a geodesic if $\gamma' \in \mathcal{V}(\gamma)$ is a parallel field, i.e. $D_t \gamma' = 0$.

In Euclidean space, these are straight lines. If $M \subset \mathbb{R}^n$ this is the condition $\pi_{TM}(\gamma'') = 0$ that we saw before. (Ex: S^2) every length-minimizing curve in M is a geodesic. (Later, we'll generalize to arbitrary manifolds.) ~~Today, basic properties.~~
Today, Basic properties.

~~Ex:~~ ① : γ has constant speed: $\frac{d}{dt} \langle \gamma' | \gamma' \rangle = \langle D_t \gamma' | \gamma' \rangle + \langle \gamma' | D_t \gamma' \rangle$

② If γ is a geodesic, $c \in \mathbb{R}$, then $\alpha(s) = \gamma(cs)$ is a geodesic. $= 0$.

$$D_s \alpha'(s) = D_s\, c\, \gamma'(cs) = c^2 D_t \gamma'(cs) = 0.$$

③ Local existence/uniqueness: $\forall x_0 \in M$ $\exists W \ni x_0$, $\varepsilon > 0$ s.t. $\forall x \in W$, $\forall v \in T_x M$ s.t. $\|v\| < \varepsilon$, $\exists!$ geodesic $\gamma_v: (-2,2) \to M$ with $\gamma_v(0) = x$, $\gamma_v'(0) = v$.

Pf: Let $\varphi = (x^1, \dots, x^n): U \to \mathbb{R}^n$ a coord. chart. Let

$$\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t)) \text{ wrt } \varphi.$$

Let $v^i = (\varphi^i)'$. Then φ is a geod $\iff$

$$0 = D_t \frac{d\gamma}{dt} = D_t [v^i \partial_i] = \frac{dv^i}{dt} \partial_i + v^i D_t[\partial_i]$$
$$= \frac{dv^i}{dt} \partial_i + v^i \nabla_{v^j \partial_j} \partial_i$$
$$0 = \frac{dv^k}{dt} \partial_k + v^i v^j \Gamma_{ij}^k \partial_k.$$
$$\frac{dv^k}{dt} = -v^i v^j \Gamma_{ij}^k \quad \forall k.$$

$\frac{dv^k}{dt} = -v^i v^j \Gamma^k_{ij}(\gamma^1, \dots, \gamma^n)$, $\frac{d\gamma^k}{dt} = v^k$. System of ODEs, so local solution exists for any initial cond, depends smoothly on initial.

That is, let ~~Let $W \subset TM$ $W = \dots$~~ W be a compact nbhd of x_0, ~~$B = \dots$~~

$K = \{(x,v) \in TM \mid x \in W, \|v\| \le 1\}$. Since K compact, $\exists\, \delta > 0$ s.t. if $(x,v) \in K$, then $\exists\, \gamma_v : (-\delta, \delta) \to M$ a geod with $\gamma_v(0) = x$, $\gamma_v'(0) = v$.

Not quite the statement but we can reparameterize.

~~For the~~ Let $\varepsilon = \frac{\delta}{4}$. For ~~all~~ $x \in V$, $v \in T_x M$, $\|v\| < \varepsilon$, $t \in (-2,2)$ let $\lambda_v(t) = \gamma_{\frac{2}{\delta}v}(\frac{\delta}{2}t)$ — then λ_v is defined on $(-2,2)$, satisfies desired conditions. //

~~Def:~~ For suff. small $p \in M$, define the exponential map $\exp_p : T_pM \to M$ by $\exp_p(v) = \gamma_v(1)$. — then $\exp_p(tv) = \gamma^p_{tv}(1) = \gamma_v(t)$ is the geodesic in the direction of v.

Many applications, which we'll explore over the semester. To start:

Application: ~~Curvature on surfaces~~ Connections on surfaces

Rethink this next time not so low, but maybe introduce geodesic curvature first, then use this as example

Let M be a Riemannian 2-manifold, $\gamma : I \to M$ a curve. How can we describe ∇ on γ? If you play around with physical surfaces, you might notice s'thing interesting: ~~there~~ if we take a nbhd of γ, that nbhd is almost flat, — turns out that you can use that to calculate the connection.

On any chart, $\varphi : U \to \mathbb{R}^n$

Recall: $\langle \nabla_{\partial_i} \partial_j \mid \partial_k \rangle = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$. If φ is an isometric embedding ~~If M is flat~~ ($g_{ij} = \delta_{ij}$), then this is zero, i.e. $\nabla_{\partial_i} \partial_j = 0$.

~~If we can choose the chart so~~

~~In general,~~ In general, this isn't possible, but

Def: We say that $f : M \to N$ is an almost isometry at $p \in M$ if ~~g~~ $\exists$ a chart $\varphi : U \to \mathbb{R}^n$ s.t. if $g_{ij} = \langle \partial_i \mid \partial_j \rangle$, $g'_{ij} = \langle \varphi_*(\partial_i) \mid \varphi_*(\partial_j) \rangle$,

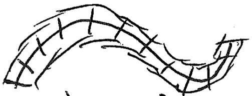
then $g_{ij}(p) = g'_{ij}(p)$, $\partial_i g_{jk}(p) = \partial_i g'_{jk}(p)$. — then ∇^M and ∇^N agree at p.

~~So, can we f~~

Claim: Let $\gamma : [0,1] \to M$ a simple curve, $\|\gamma'(t)\| = 1$. Then $\exists$ a map $\varphi : U \to \mathbb{R}^2$, ~~an almost isometry~~ ~~from~~ $U \supset \gamma$ s.t. φ is an almost-isom on γ.

(Potato peel lemma: if you peel a potato, you get a flat strip — what's that strip?)

Let ~~$N \in \mathcal{V}(\gamma)$~~ $V = \gamma' \in \mathcal{V}(\gamma)$, $N \in \mathcal{V}(\gamma)$ orthonormal to V:



Let $\alpha : [0,l] \times (-\varepsilon, \varepsilon) \to M$

$\alpha(t,u) = \exp_{\gamma(t)}(uN(t))$.

Let $\partial_u = \frac{\partial \alpha}{\partial u}$, $\partial_t = \frac{\partial \alpha}{\partial t}$. vector fields on α.

Let $g_{tt} = \langle \partial_t | \partial_t \rangle$
$g_{tu} = \langle \partial_t | \partial_u \rangle$
$g_{uu} = \langle \partial_u | \partial_u \rangle$.

Let $M = \begin{pmatrix} g_{tt} & g_{tu} \\ g_{ut} & g_{uu} \end{pmatrix}$.

When $u = 0$, ∂_u, ∂_t orthonormal $\Rightarrow M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\partial_t M = 0$.

~~Calculate~~ $\|\partial_u\| = 1$ $\forall u, t$, so $g_{uu} = 1 \Rightarrow \partial_t g_{uu} = 0$.

$$\partial_u g_{tu} = \partial_u \langle \partial_t | \partial_u \rangle = \langle D_u \partial_t | \partial_u \rangle + \langle \partial_t | \underbrace{D_u \partial_u}_{\to 0 \text{ (geodesic)}} \rangle$$

$$\tau(\partial_t, \partial_u) = D_t \partial_u - D_u \partial_t - \underbrace{[\partial_t, \partial_u]}_{\to 0}$$

$$= \langle D_t \partial_u | \partial_u \rangle = \tfrac{1}{2} \partial_t \langle \partial_u | \partial_u \rangle = 0.$$

So when $u = 0$, $\partial_t M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\partial_u M = \begin{pmatrix} \partial_u g_{tt} & 0 \\ 0 & 0 \end{pmatrix}$ Let $\kappa = -\partial_u g_{tt}$.

Let $\theta(t) = \int_0^t \kappa \, d\tau$, let $\bar{\gamma}(t) = \int_0^t e^{i\theta(\tau)} d\tau$

let $\bar{\alpha} : [0, l] \times (-\varepsilon, \varepsilon) \to \mathbb{R}^2$

$$\bar{\alpha}(t,u) = \bar{\gamma}(t) + u i e^{\theta(t)}$$

Check: $\partial_t \bar{M} = 0$
$\partial_u \bar{M} = \begin{pmatrix} -\kappa & 0 \\ 0 & 0 \end{pmatrix}$ — so $\varphi = \bar{\alpha} \circ \alpha^{-1}$ is the desired map.

~~Because~~ Then ∇^M agrees with $\nabla^{\mathbb{R}^2}$ on γ: if $W \in \mathcal{V}(\gamma)$, then

$$D_t^{\mathbb{R}^2}[\varphi_*(W)] = \varphi_*[D_t^M W].$$

W is parallel in M $\Leftrightarrow$ $\varphi_*(W)$ is parallel in $\mathbb{R}^2$
$\Leftrightarrow$ $\varphi_*(W)$ is constant.

So we can construct a parallel frame on γ by:
- map nbhd of γ to $\mathbb{R}^2$
- draw a constant frame on the nbhd
- pull back.

Overflow: illustrate ~~holonomy~~, geodesics, holonomy.

Maybe next time: Define geodesic curvature, ~~2~~ 2025-02-18
parallel fields in terms of geod curvature, then explain in terms of potato peels.

Last time:

Surfaces: In particular, nbhd of a curve:

Lemma: Let $\gamma : [0,1] \to M$ a simple curve, $\dim M = 2$, $\|\gamma'\| = 1$.
Then $\exists$ a map $\varphi : U \to \mathbb{R}^2$ s.t. φ is an almost isom on γ.
~~Fact~~

(i.e., for any vector fields $X, Y \in \mathcal{V}(U)$, the fn ~~$\|V\|_M$ and $\|\varphi_*(V)\|_{\mathbb{R}^2}$~~

$$E(p) = \langle X_p | Y_p \rangle - \langle \varphi_*(X_p) | \varphi_*(Y_p) \rangle$$

vanishes to first order on γ. $\Leftrightarrow \exists$ chart s.t. $\partial_i g_{jk} = 0$ on γ $\forall i,j,k$.

~~Further, for any such map $\varphi(\gamma)$~~ is If φ, ψ are two such maps, then $\varphi(\gamma)$ and $\psi(\gamma)$ agree up to an isometry of $\mathbb{R}^2$.

Pf: Let $V = \gamma' \in \mathcal{V}(\gamma)$, $N \in \mathcal{V}(\gamma)$ s.t. ~~V, N are~~ orthonormal to V.

$\alpha : [0,1] \times (-\varepsilon, \varepsilon) \to M$

$\alpha(t,u) = \exp_{\gamma(t)}(uN)$ Let $\partial_u = \frac{\partial \alpha}{\partial u}$, $\partial_t = \frac{\partial \alpha}{\partial t} \in \mathcal{V}(\alpha)$ be std basis.

Let
$g_{tt} = \langle \partial_t | \partial_t \rangle$
$g_{tu} = \langle \partial_t | \partial_u \rangle$, $M = \begin{pmatrix} g_{tt} & g_{tu} \\ g_{tu} & g_{uu} \end{pmatrix}$
$g_{uu} = \langle \partial_u | \partial_u \rangle$

How do these behave? when $p \in \gamma$, ~~$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$~~
∂_u, ∂_t orthonormal $\Rightarrow M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\Rightarrow \partial_t M = 0$ when $u = 0$

~~so $\partial_t g_{tt} = \partial_t g_{tu} = \partial_t g_{uu} = 0$ on γ.~~

$\partial_u g_{uu} = \partial_u [\|\partial_u\|^2] = \partial_u[1] = 0$ because $u \mapsto \exp_{\gamma(t)}(uN)$ is a geodesic of unit speed.

$\partial_u g_{ut} = \partial_u \langle \partial_u | \partial_t \rangle = \underbrace{\langle D_u \partial_u | \partial_t \rangle}_{0} + \langle \partial_u | D_u \partial_t \rangle$

$0 = \tau(\partial_u, \partial_t) = D_u \partial_t - D_t \partial_u - \underbrace{[\partial_u, \partial_t]}_{0}$

so
$\partial_u g_{ut} = \langle \partial_u | D_t \partial_u \rangle = \frac{1}{2} \partial_t \langle \partial_u | \partial_u \rangle = 0$

So $\partial_u M = \begin{pmatrix} \partial_u g_{tt} & 0 \\ 0 & 0 \end{pmatrix}$ — and from the picture, we expect that $\partial_u g_{tt}$ varies depending on how γ turns.

~~indeed, $\partial_u \langle \partial_t | \partial_t \rangle = 2 \langle D_u \partial_t | \partial_t \rangle = 2 \langle D_t \partial_u | \partial_t \rangle$~~

Map to $\mathbb{R}^2$ the same way: let $\kappa(t) = -\frac{1}{2} \partial_u g_{tt}(t)$.

$\theta(t) = \int_0^t \kappa(\tau) \, d\tau$, let $\bar{\gamma}(t) = \int_0^t e^{i\theta(\tau)} \, d\tau$, let

$$\bar{\alpha} : [0,1] \times (-\varepsilon, \varepsilon) \to \mathbb{C} \quad \bar{\alpha}(t,u) = \bar{\gamma}(t) + u i e^{i\theta(t)}$$

Then $\partial_t \bar{M} = 0$, $\partial_u \bar{M} = \begin{pmatrix} -2\kappa & 0 \\ 0 & 0 \end{pmatrix} = \partial_u M$, so $\bar{\alpha} \circ \alpha^{-1}$ is desired map.

Exercise: ~~$\bar{\alpha}$ is unique~~ $\bar{\alpha}|_{[0,1] \times \{0\}}$ is unique //

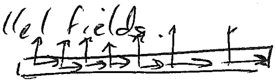
Since φ is almost-isom on γ, ∇^M and $\nabla^{\mathbb{R}^2}$ agree on γ, i.e. $\forall W \in \mathcal{V}(\gamma)$,

$$D_t^M W = \varphi_*^{-1}\left(D_t^{\mathbb{R}^2}[\varphi_* W]\right).$$
$$= \varphi_*^{-1}\left(\tfrac{d}{dt}\varphi_*(W)\right)$$

In particular, W parallel $\Leftrightarrow$ $\varphi_*(W)$ is constant!

we can construct a parallel field by taking a curve, cutting out a nbhd, laying flat, taking a const field on flattened curve.

So we can use that to get ~~some~~ a feel for parallel fields.



Ex: parallel fields on a geodesic.

Ex: holonomy: closed curves don't flatten.

$\Rightarrow$ V rotates relative to parallel field.

~~Define this?~~

~~Let γ be a unit-speed curve.~~

~~For a unit-speed curve γ, let $V = \gamma' \in \mathcal{V}(\gamma)$, let N~~

~~Essentially as in~~ Can define this as in lemma: Let $\gamma : [0,\ell] \to M$, $\dim(M) = 2$, let $V = \gamma'$, N orth. to V. Then

$$\frac{d}{dt}\langle V | V\rangle = 0 = 2\langle D_t V | V\rangle, \text{ so } D_t V \text{ is orthogonal to } V.$$

$\Rightarrow \exists\, \kappa : [0,\ell] \to \mathbb{R}$ s.t. $D_t V = \kappa N$. — call κ the geodesic curvature of γ.

If $\gamma : [0,\ell] \to \mathbb{R}^2 \cong \mathbb{C}$, then $\gamma' = e^{i\theta(t)}$

$$D_t\gamma' = \gamma'' = i\theta'(t)e^{i\theta(t)} = \theta'(t)N(t).$$
$$\Rightarrow \kappa = \theta'(t).$$

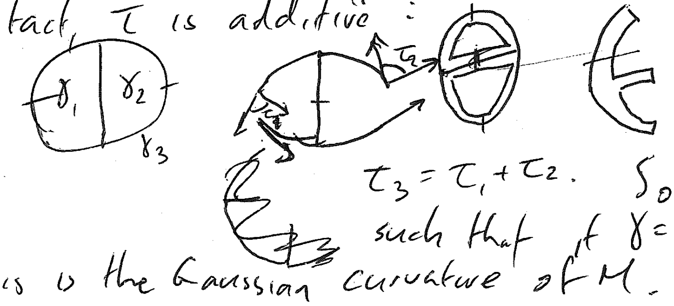
If $\gamma : S^1 \to \mathbb{R}^2$ is positively oriented simple closed curve, then

$$\int_{S^1}\kappa\, dt = 2\pi$$

If $\gamma : S^1 \to S^2$?

For γ a unit-speed closed curve, Let $\tau(\gamma) = 2\pi - \int_\gamma \kappa\, dt$. then τ ranges from 0 to 4π. here.

In fact, τ is additive:



$\tau_3 = \tau_1 + \tau_2$. So $\exists$ a fn $K \in C^\infty(M)$ such that if $\gamma = \partial D$, then $\int_\gamma \kappa\, dt = \int_D K\, dA$.

This is the Gaussian curvature of M.

On the sphere, this is constant — equator is geodesic $\kappa = 0$, bounds hemisphere

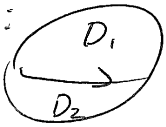
$\tau(\text{equator}) = 2\pi - 0 = 2\pi$, $K = \dfrac{\tau(\text{equator})}{A(\text{hemisphere})} = \dfrac{2\pi}{2\pi r^2} = \dfrac{1}{r^2}$.

On a plane, every simple closed curve has $\tau = 0$.

On a saddle surface, $\tau < 0$, negative curvature

Thus: Thm (Gauss-Bonnet): Let $M \cong S^2$. Then $\int K\,dA = 4\pi$.

Pf: Let γ cut M into discs D_1, D_2. Then:



$$\tau(\gamma) = 2\pi - \int_\gamma \kappa\,dt = \int_{D_1} K\,dA$$

$$\tau(-\gamma) = 2\pi - \int_{-\gamma} \kappa\,dt = 2\pi + \int_\gamma \kappa\,dt = \int_{D_2} K\,dA$$

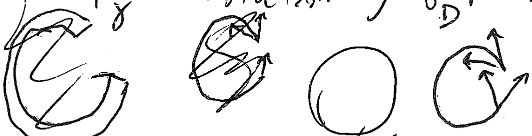
$$\int_M K\,dA = \int_{D_1} K\,dA + \int_{D_2} K\,dA = 4\pi \quad //$$

Exercise: For compact surf Σ, suppose Σ has a triang with F faces, E edges, V verts. $\chi(\Sigma)$

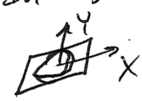
Thm (Gauss-Bonnet): For any compact orientable surface Σ, $\int_\Sigma K\,dA = 2\pi\chi(\Sigma)$ ~~$(F-E+V)$~~
where $\chi(\Sigma) = F - E + V$ for ~~some~~ any triangulation of Σ.

~~(May also have seen~~ <u>Higher dimensions?</u>: ~~Need more.~~

Another expression: For $\gamma : [0,l] \to M$ a ~~closed~~ curve, ~~the~~ let $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(l)}M$ be parallel transport.
If γ is closed curve with $\gamma(0) = \gamma(l) = p$, then $P_\gamma : T_pM \to T_pM$ is called the holonomy of M. In 2D, if $\gamma = \partial D$, then
P_γ = rotation by $\int_D K\,dA$.



In higher dims, what?



Curves could have different orientations.

rotations ~~could be more complex~~ are $n\times n$ matrices. Need: ~~Curvature tensor~~

Def For connection ∇, the <u>curvature tensor</u> of ∇ is $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ s.t.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Then: ① R is alternating in X and Y. ② R is trilinear over $\mathbb{R}$.

Lemma: $(R(X,Y)Z)_p$ depends only on X_p, Y_p, Z_p.

Pf: Let $f \in C^\infty(M)$. Check:

$$R(X,fY)Z = \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X,fY]} Z$$

$$= \nabla_X[f \nabla_Y Z] - f\nabla_Y \nabla_X Z - f\nabla_{[X,Y]} Z - (Xf)\nabla_Y Z$$

$$= \cancel{(Xf)\nabla_Y Z} + \cancel{f\nabla_X \nabla_Y Z} + fR(X,Y)Z - \cancel{(Xf)\nabla_Y Z}$$

and likewise for $R(X,Y)[fZ] = f R(X,Y)Z$.

So if $X = f^i \partial_i$, $Y = g^j \partial_j$, $Z = h^k \partial_k$,

$$R(X,Y)Z = f^i g^j h^k R(\partial_i, \partial_j)\partial_k$$

$$(R(X,Y)Z)_p = f^i(p) g^j(p) h^k(p) R(\partial_i, \partial_j)\partial_k \quad //$$

Parallel transport and Gauss-Bonnet: Let M be a connected oriented 2-manifold.

~~Every (Given any chart $\varphi: U \to \mathbb{R}^2$~~, you can assign an

(~~given any p o.n. frame~~ $p \in M$, any o.n basis ~~of~~

~~(the set of o.n. bases~~ (Let $G = \{$o.n bases of fibers of $TM\}$ — then G has two connected cpts, which we call positively/negatively oriented)

Equiv, there's a well-defined notion of CW/CCW rotation — let $R_\theta: TM \to TM$ be CCW rotation.

Let $\gamma: [0, l] \to M$ be a unit-speed curve, let $V = \gamma'$, let $N = R_{\frac{\pi}{2}}(\gamma')$. ~~Then $D_t V$ is orthogonal to V, so~~

Let $\kappa = \langle N, D_t V \rangle$. Then $D_t V$ is orthog to V, so $D_t V = \kappa N$. Call κ the geodesic curvature of γ.

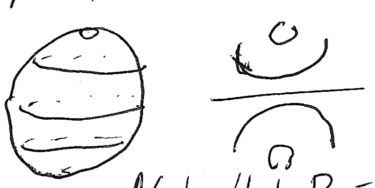
Then κ measures turning: two ways:

— Exercise: Let $\theta(t) = \int_0^t \kappa(\tau)\, d\tau$. Then ~~$R_{-\theta(t)}(\gamma'(t))$~~ a ~~parale~~ $R_{-\theta(t)}(\gamma'(t))$ is a parallel field.

— Last time: $\exists$ an almost isom $\gamma \subset U \to \mathbb{R}^2$ taking γ to $\gamma(t) = \int_0^t e^{i\theta(\tau)}\, d\tau$.

— Holonomy: So $P_\gamma = R_{-\int_0^l \kappa(t)\, dt}$.

Paper strips gives us a way to see what that means:



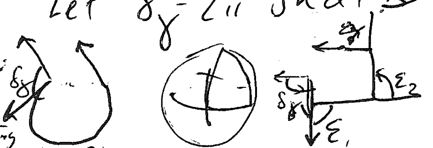
$\int_0^l \kappa\, dt = 2\pi - \varepsilon$
$\approx \pi$
≈ 0
$\approx -\pi$
$\approx 2\pi + \varepsilon$

$2\pi - \int \kappa\, dt \approx$ area.

Let $\delta_\gamma = 2\pi - \int \kappa\, dt = \sum \varepsilon_i$

— Note that $P_\gamma = R_{\delta_\gamma}$.

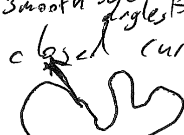
Note:



~~Lemma: $2\pi - \int \kappa\, dt$ is additive~~

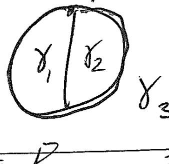
Then: Extend to piecewise smooth by considering angles ~~Extend to PL curves~~

Lemma: if $\gamma: S^1 \to \mathbb{R}^2$ is a closed curve, then $\delta_\gamma = 0$



~~If we handle corners properly,~~

Lemma: Then δ_γ is additive: if

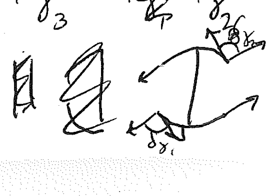


then $\delta_{\gamma_3} = \delta_{\gamma_1} + \delta_{\gamma_2}$

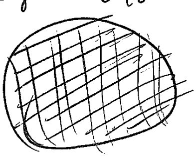
1: ~~Clear mod 2π:~~ $P_\gamma = R_{-\int \kappa\, dt} = R_{2\pi - \int \kappa\, dt}$

$P_{\gamma_3} = P_{\gamma_2} \circ P_{\gamma_1} \Rightarrow \delta_{\gamma_1} + \delta_{\gamma_2} \equiv \delta_{\gamma_3} \pmod{2\pi}$

2: Picture:



So δ_γ should act like an integral:— $\exists\, K \in C^\infty(M)$ s.t. if $\gamma = \partial D$ traced with positive orientation,



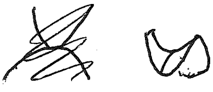
then $\delta_\gamma = \int_D K\,dA$ — this K is the <u>Gaussian curvature</u> of M.

Then: $M = \mathbb{R}^2$, $\delta_\gamma = 0$ $\forall$ simple closed curves $\Rightarrow K = 0$.

$M = S^2(r)$, ~~sphere of rad~~ let γ be equator, $\delta_\gamma = 2\pi - \int \kappa\, dt = 2\pi$.

$K = \frac{2\pi}{(4\pi r^2)/2} = \frac{1}{r^2}$.

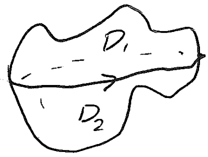
~~How~~ $K < 0$?



Saddle points of surfaces in $\mathbb{R}^3$.

Thm (Gauss-Bonnet): Let $M \cong S^2$. Then $\int K\, dA = 4\pi$.

Pf: Let γ cut M into discs D_1, D_2:



Then $\delta_\gamma = 2\pi - \int_\gamma \kappa\, dt = \int_{D_1} K$.

But also, $\delta_{-\gamma} = 2\pi - \int_{-\gamma} \kappa\, dt = 2\pi + \int_\gamma \kappa\, dt = \int_{D_2} K$.

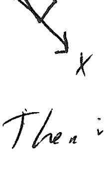
So $\delta_\gamma + \delta_{-\gamma} = \int_{S^2} K = 4\pi + \int_\gamma \kappa\, dt - \int_\gamma \kappa\, dt = 4\pi$.

More generally, Thm: Let M be an oriented 2-manifold, with a triangulation with F faces, E edges, V vertices. Then $\int_M K\, dA = 2\pi(F - E + V)$.

Q: How can we calculate K? How can we generalize to higher dims?

~~Curvature tensor.~~ ~~This is a lot more complex~~ This is more involved in higher dims: in dim 2, this is parallel transport around a loop, and because $SO(2) \cong S^1$, just rotation.

In higher dims, need sthing like:



So we define: $R : \mathcal{V}(M) \times \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M)$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \quad \text{Then:}$$

① $R(X,Y)Z = -R(Y,X)Z$.

② R is trilinear over $\mathbb{R}$.

Lemma: R is a tensor — $(R(X,Y)Z)_p$ depends only on X_p, Y_p, Z_p.

Pf: Let $f \in C^\infty(M)$. Check:

$$R(X,fY)Z = \nabla_X \nabla_{fY} Z - \nabla_{fY}\nabla_X Z - \nabla_{[X,fY]} Z$$

$$= \nabla_X[f\nabla_Y Z] - f\nabla_Y\nabla_X Z - \nabla_{(Xf)Y + f[X,Y]} Z$$

$$= \cancel{(Xf)\nabla_Y Z} + f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - \cancel{(Xf)\nabla_Y Z} - f\nabla_{[X,Y]} Z$$

$$= fR(X,Y)Z$$

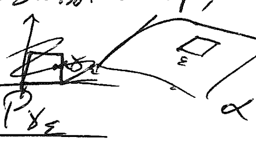
So if $X = x^i \partial_i$, $Y = y^j\partial_j$, $Z = z^k\partial_k$, then $R(X,Y)Z = x^i y^j z^k R(\partial_i,\partial_j)\partial_k$;
$(R(X,Y)Z)_p = x^i(p) y^j(p) z^k(p) R(\partial_i,\partial_j)\partial_k$

R depends only on X_p, Y_p, Z_p. //. This ~~describes parallel holonomy of short curves around~~

~~Thm (Ambrose-Singer):~~

(1) This describes holonomy around short curves:

Thm (Ambrose-Singer): ~~Let~~ $U \to M$,
Let $0 \in U \subset \mathbb{R}^2$, let $\alpha : U \to M$ be a smooth map, $\alpha(0) = p$,
$X_p = \frac{\partial\alpha}{\partial x}\big|_0$ $Y_p = \frac{\partial \alpha}{\partial y}\big|_0$. Let $\gamma_\varepsilon = \partial(\alpha[0,\varepsilon]^2)$



Then $\forall Z_p \in T_pM$, $R(X,Y)Z = \lim_{\varepsilon\to 0} \frac{Z_p - P_{\gamma_\varepsilon}}{\varepsilon^2}$

(2) So this ~~describes exist~~ obstructs parallel fields.
Suppose $U \subset M$, Z is parallel on U, i.e. $\nabla_X Z = 0 \ \forall X$.

Then $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z = 0$
Furthermore, by symmetry, $R(Z,X)Y = R(Y,Z)X = 0$,
~~so R vanishes on Z~~ and $\langle R(X,Y)W | Z\rangle = 0$

(3) Prop: R satisfies some symmetries: $\forall X,Y,Z,W \in \mathfrak{X}(M)$.
(1) $R(X,Y)Z = R(Y,X)Z$.
(2) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.
(3) $\langle R(X,Y)Z | W\rangle = -\langle R(X,Y)W | Z\rangle$
(4) $\langle R(X,Y)Z|W\rangle = \langle R(Z,W)X | Y\rangle$.

Conversely: ~~Thm: $R \equiv 0 \iff M$ is locally isometric to $\mathbb{R}^n$~~
Lemma: ~~$R\equiv 0$~~ Suppose $R \equiv 0$. Then $\forall p \in M$ $\forall X_p \in T_pM$, there is a nbhd $U \ni p$ s.t. X_p extends to a parallel field on U.

Overflow: questions

2025-02-24

Last time: Riemann curvature tensor:
$\forall X, Y, Z \in \mathcal{V}(M)$, $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

Symmetries ~~Properties~~: $\forall X, Y, Z, W \in \mathcal{V}(M)$

① $R(X,Y)Z = -R(Y,X)Z$ ② $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

③ $\langle R(X,Y)Z | W \rangle = \langle R(Z,W)X | Y \rangle$

④ $\langle R(X,Y)Z | W \rangle = -\langle R(X,Y)W | Z \rangle$.

Ex: Check these.

Measures holonomy: Thm (Ambrose-Singer): Let $0 \in U \subset \mathbb{R}^2$, let $\alpha: U \to M$ a smooth map, let $p = \alpha(0)$, $X_p = \frac{\partial \alpha}{\partial x}(0)$, $Y_p = \frac{\partial \alpha}{\partial y}(0)$.

Let $\gamma_\varepsilon = \alpha(\partial[0,\varepsilon]^2)$. Then $\forall Z_p \in T_pM$, $R(X,Y)Z = \lim_{\varepsilon \to 0} \frac{Z_p - P_{\gamma_\varepsilon}(Z_p)}{\varepsilon^2}$

~~So given a curv~~ Pf: Exercise.

What does curvature measure? ① ~~(Last time)~~ obstruction: ~~existence~~ ~~of~~ parallel fields $\}$.

~~② Obstruction to flatness~~: ~~More concretely~~:

Last time: if $X \in \mathcal{V}(U)$ is parallel ($\nabla_Y X = 0 \ \forall Y$), then $R(X,Y)Z = R(Y,Z)X = 0$, $\langle R(Y,Z)W | X \rangle = 0 \ \forall W, Y, Z$.

Conversely:

Lemma: Suppose $R \equiv 0$. Then $\forall p \in M$, $\forall V_p \in T_pM$, $\exists$ a nbhd $U \ni p$ and ~~$W \in \mathcal{V}(U)$ s.t. $W_p = V_p$~~ a parallel field $W \in \mathcal{V}(U)$ s.t. $W_p = V_p$.

Pf: Let $\varphi = (u^1, \dots, u^n): U \to B \subset \mathbb{R}^n$ be a chart s.t. B = unit ball, $\varphi(p) = 0$. We write $(x^1, \dots, x^n)$ for $\varphi^{-1}(x^1, \dots, x^n) \in U$.

For $q = (q^1, \dots, q^n)$, let γ_q be the path $(0, \dots, 0) - (q^1, 0, \dots, 0) - (q^1, q^2, 0, \dots, 0) - (q^1, \dots, q^n)$. Let $W_q = P_{\gamma_q}(V_p)$. Then $W \in \mathcal{V}(U)$.

Let $U_i = u_1 \cdots u_i$-plane. Claim W is parallel along $U_i \ \forall i$.

$i = 0$: By construction. Suppose W is parallel along U_{i-1}:

∂_i U_i We construct W on U_i by parallel transport in ∂_i dir, so $\nabla_{\partial_i} W = 0$ on U_i.

Let $j < i$. Then $\nabla_{\partial_j} W = 0$ on U_{i-1} ~~for $j < i$~~ by induction. Claim $\nabla_{\partial_j} W = 0$ on U_i. We have

$$R(\partial_i, \partial_j) W = \nabla_{\partial_i} \nabla_{\partial_j} W - \nabla_{\partial_j} \nabla_{\partial_i} W - \nabla_{[\partial_i, \partial_j]} W = 0$$

$\Rightarrow \nabla_{\partial_i} \nabla_{\partial_j} W = 0$ on U_i

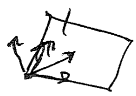
$\Rightarrow \nabla_{\partial_j} W$ is parallel along ∂_i-lines – but $\nabla_{\partial_j} W = 0$ on U_{i-1},

so $\nabla_{\partial_j} W = 0$ on all of U_i. //

② Obstruction from flatness:
Thm: ~~Pf~~ Suppose $R \equiv 0$. Then $\forall p \in M$, $\exists U \ni p$ s.t. U is isometric to a subset of $\mathbb{R}^n$.

Pf: Let $E_1, \dots, E_n \in T_pM$ o.n. basis. By lemma, $\exists$ $U_0 \ni p$, parallel fields $W_1, \dots, W_n \in \mathfrak{X}(U_0)$ extending E_i, and these are orthonormal. ~~Then~~ By exercise, torsion measures whether parallel fields commute, so
$$0 = \tau(W_i, W_j) = \nabla_{W_i} W_j - \nabla_{W_j} W_i - [W_i, W_j]$$
but W_i, W_j parallel $\Rightarrow [W_i, W_j] = 0$. By Frobenius Lemma $\exists B \subset' \mathbb{R}^n$, $\alpha : B \to \mathbb{R}^n$ s.t. $\alpha(0) = p$, $\frac{\partial \alpha}{\partial x^i} = \alpha_*\left(\frac{\partial}{\partial x^i}\right) = W_i$.
So α takes o.n. basis in $\mathbb{R}^n$ to an o.n. basis in M — preserves metric //.

③ ~~Geom~~ Divergence of geodesics: We don't have all the tools to describe this yet, but:
Def: For $p \in M$, $X, Y \in T_pM$, let $K(X,Y) = \dfrac{\langle R(X,Y)Y \mid X\rangle}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}$
(in particular, if X, Y o.n., then $K(X,Y) = \langle R(X,Y)Y \mid X\rangle$.)

When $\dim M = 2$, this is ~~see~~ Gaussian curvature:  how much does Y rotate?

~~Prop~~ Prop: ① If $\mathrm{span}(X,Y) = \mathrm{span}(U,V)$, then $K(X,Y) = K(U,V)$.
~~Pf: Exercise~~
② The Riemann curvature tensor is determined by K.
Pf: Exercise.
Thm: $\forall n \geq 2$, $\forall K \in \mathbb{R}$, $\exists$ a unique geodesically complete (any geodesic can be extended infinitely), simply-connected model space M^n_K s.t. $\forall p \in M^n_K$, $\forall X, Y \in T_pM$, $K(X,Y) = K$.
- if $K > 0$, $M^n_K = S^n\left(\frac{1}{\sqrt{K}}\right)$ n-sphere.
- if $K = 0$, $M^n_K = \mathbb{R}^n$
- if $K < 0$, $M^n_K = \mathbb{H}^n$ hyperbolic n-space.

(which you've seen: problem set, $dg^2 = \dfrac{dx^2 + dy^2}{y^2}$, but also the Poincaré disk model:
$$M^2_{-1} = \mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \quad dg^2 = \frac{4(dx^2 + dy^2)}{(1 - (x^2+y^2))^2} = \frac{4(dx^2+dy^2)}{(1-r^2)^2}$$

This is radially symmetric, so by the problem set, ~~any diameter is a geodesic~~ the geodesics through the origin are the diameters of the circle — ~~these are infinite length~~ these have length $\int_{-1}^{1} \frac{1}{1-r^2}\,dr = \tanh^{-1} r \Big|_{-1}^{1} = \infty$
~~And~~ Looks possibly incomplete, but:
~~And we can calculate arc~~

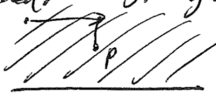
~~And then~~ In fact, ~~Isom~~ $\forall n \geq 2$, $K \in \mathbb{R}$,

Isom M^n_K acts transitively on ~~$T M^n_K$~~ UM^n_K = unit tangent bundle.

~~Clear for S^n, $\mathbb{R}^n$. For $\mathbb{H}^2$, can~~

~~For $\mathbb{R}^n$~~, ~~Clear~~ Familiar for S^n, $\mathbb{R}^n$. For $\mathbb{H}$, follows from Pset – ~~fractional linear~~ or you can construct: two models:

~~Clearly~~,



$\left(U, \frac{dx^2+dy^2}{y^2}\right)$

$\left(D, \frac{4}{(1-r^2)^2} dx^2+dy^2\right)$

$$z \overset{\varphi}{\longmapsto} \frac{z-i}{z+i}$$

$\forall$ ~~a, b~~ $a>0$, $b \in \mathbb{R}$,

$\tau_b(z) = z+b$ ~~$\alpha_{a,b}(z) = az+b$ is a~~

$\alpha_a(z) = az$ are isometries

– so Isom($\mathbb{H}$) acts trans. on M

$\forall \theta \in [0, 2\pi]$

$R_\theta(z) = e^{i\theta}z$ is an isometry.

– so act trans on ~~T_pM~~.

~~$\forall w \in U, \exists a, b$ st $(\alpha_a(\tau_b))(w) = i$~~ ~~So~~ ~~$S_p$~~ ~~Isom~~ let $v \in T_p$~~M~~ U_pM

Then Isom($\mathbb{H}$)$\cdot v \supseteq U_pM$ (rotations).

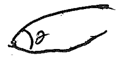
Isom($\mathbb{H}$)$\cdot v = U_pM$. (translations).

~~We don't have tools yet to prove that these are the only model spaces – so next we work on that:~~ ~~How does~~ ~~Geodesics~~ ~~and~~

~~So what can we say?~~ Divergence of geodesics:

~~Prop: Let $n \geq 2$, $K \in \mathbb{R}$, $p \in M^n_K$. Let $V, W \in T_pM^n_K$, $\angle(V, W) = \theta$.~~

~~Calculate~~ ~~V, W~~ K affects lengths of arcs:



$K<0$

arclen $= \theta\sqrt{-K} \sinh\left(\frac{r}{\sqrt{-K}}\right)$

$K=0$

arclength $r\theta$

$K>0$

arclen $=$ ~~$\theta\sqrt{K}\sin($~~

$\theta\sqrt{K}\sin\left(\frac{r}{\sqrt{K}}\right)$

– ~~same~~ close at small r, or small K ~~close at~~ ~~small~~ ~~large~~ K but ~~not the same~~ larger $K \Rightarrow$ closer geods.

(Break, questions?)

We don't have tools to prove this yet, ~~but still~~ so next:

Geodesics and ~~normal coordinates~~:

Recall: $\forall p \in M$ $\exp_p :$ ~~$\mathbb{R}^n$~~ $T_pM \to M$ is the map s.t.

$\forall v \in T_pM$, $t \mapsto \exp_p(tv)$ is the geod through p ~~in the direction~~ with veloc. v. –

~~This is locally a diffeo~~ ~~We call~~

~~Then~~ ~~$\exp$~~ ~~$(D\exp_p)_0$ is nonsingular, so $\exp$~~

Then $D_0\exp_p$ is nonsingular, so $\exp_p$ is locally a diffeo – we call the corresp chart a normal coordinate chart on ~~$\mathbb{R}$~~ M –

– How does this behave?

– ~~Though~~ How big of a chart can we construct – what points can we connect by geodesics?

2025-02-26

Geodesics:

Prev: $\gamma : I \to M$ is a geodesic if $D_t \gamma' = 0 \; \forall t \in I$.

~~$\exp_p(v) = \gamma_v(1)$, where γ_v is geod with $\gamma_v(0) = p$, $\gamma_v'(0) = v$~~

Two things: ~~(1) When is a geodesic length minimizing?~~

(1) When are two points connected by a geodesic?

(2) When is a geodesic length-minimizing?

(1): For any $p \in M$, there's a nbhd of p connected to p:

$\exp_p(v) = \gamma_v(1)$, where γ_v is geod with $\gamma_v(0) = p$, $\gamma_v'(0) = v$. Then

— $\exp_p : U \subset T_pM \to M$, where U is a nbhd of 0.

— $D_0 \exp_p = id_{T_pM}$, so ~~$\exp_p$ is a diffeo around~~ local diffeo at 0 — $\exists$ $0 \in U \subset T_pM$ s.t. $\exp_p$ takes U diffeo to $\exp_p(U) \ni 0$ (a normal neighborhood of p). Often, we want sthing better:

Thm (Uniformly normal nbhd thm): $\forall p \in M$, $\exists W, \varepsilon$ s.t.:

(1) $\forall x, y \in W$, x and y are connected by a unique geodesic $\gamma_{x,y}$ of length $< \varepsilon$.

(2) γ depends smoothly on x, y

(3) $\forall x \in W$, $\exp_x$ sends $B_\varepsilon(0)$ diffeo'ly to a nbhd U_x of x s.t. $W \subset U_x$.

Pf: Let $F : TM \to M \times M$, $F(x,v) = (x, \exp_x(v))$. Then F is smooth on a nbhd of $(p, 0)$. ~~We parameterize~~ Pick a chart around p, parameterize TM by $(x^1, \dots, x^n, v^1, \dots, v^n) \mapsto ((x^1, \dots, x^n), v^i \partial_i)$

Then $F(\vec{x}, 0) = (\vec{x}, \vec{x}) \Rightarrow F_*(\partial_{x^i}) = (\partial_i, \partial_i)$

$F(\vec{x}, \vec{v}) = (\vec{x}, \exp_{\vec{x}} \vec{v})$ $\quad F_*(\partial_{v^i}) = (0, \partial_i)$, i.e.

$$DF_{(p,0)} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \text{ is invertible.}$$

Choose U, ε s.t. F is invertible on U_ε

~~Let~~ $U_\varepsilon = \{(u,v) \in TM \mid u \in U, \|v\| < \varepsilon\}$, ~~where~~ Then $F(U_\varepsilon) \ni (p,p)$

choose $W \ni p$ s.t. $W \times W \subset F(U_\varepsilon)$.

Check: (1): $\forall x, y \in W$, $\exists v$ s.t. ~~$F(x,v)$~~ $(x, v) \in U_\varepsilon$, $F(x,v) = (x,y)$

$\Rightarrow \exp_x v = y$, $\|v\| < \varepsilon$.

(2) ✓ (3) $\{x\} \times B_\varepsilon(0) \subset U_\varepsilon$, so $\exp_x$ sends $B_\varepsilon(0)$ diff'ly

So sufficiently close x, y are connected by a unique length $< \varepsilon$ shortest geodesic.

Is this the shortest curve from x to y? Def: For piecewise smooth $\gamma : I \to M$, let $\ell(\gamma) = \int_I \|\gamma'(t)\| dt$. For $x, y \in M$, let

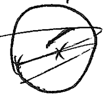
$$d(x,y) = \inf\{\ell(\gamma) : \gamma \text{ piecewise smooth}, \gamma(0) = x, \gamma(1) = y\}$$

Clearly satisfies symmetry, triangle inequality. Not a priori clear that it satisfies positivity.

Follows from this ~~thm~~:

Thm: Let $x \in M$, let $r_0 > 0$ s.t. $\exp_x$ is a diffeo on $B_{r_0}(0)$. Let $W = B_{r_0}(0)$. Let $y \in W$, let γ be the unique geod of length $< r_0$ from x to y. Then $d(x,y) = \ell(\gamma)$ and γ is the unique shortest path from x to y. Conversely, if $d(x,z) < r_0$, then $z \in W$ (so there's a unique geod from x to z of length $d(x,z)$).

~~Inside W, dist to x is realized by geods.~~

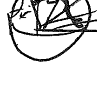
~~S_0~~  W ~~- inside W, distance to x is realized by geodesics~~

~~Any point close enough to x is in W.~~ In particular, if $d(x,z) = 0$, then $x = z$.

Gauss's Lemma: Let W, r_0 as above, let $0 < r < r_0$. Let $S_r = \{v \in T_xM \mid \|v\| = r_0\}$, so $\exp_x(S_r)$ is a sphere embedded in W. Then $\forall v \in T_xM$, γ_v is orthogonal to $\exp S_r$.

Pf: ~~ETS that $\forall v$~~ ETS that $\forall v \in S^*$, $\theta : (-1,1) \to S$, s.t. $\theta(0) = v$, if $\theta_r(t) = \exp_x(r\theta(t))$, then ~~$\theta_r'$ intersects γ_v~~ $\langle \theta_r'(0) \mid \gamma_v'(r) \rangle = 0$.

Let $f(\rho, t) = \exp(\rho\theta(t))$. Then



$$\theta_r(t) = f(r,t) \qquad \theta_r'(t) = \frac{\partial f}{\partial t}(r,t)$$

$$\gamma_v(\rho) = f(\rho, 0) \qquad \gamma_v'(\rho) = \frac{\partial f}{\partial \rho}(r, 0) \quad \text{and } \left\|\frac{\partial f}{\partial \rho}\right\| = 1$$

Consider $\langle \frac{\partial f}{\partial \rho} \mid \frac{\partial f}{\partial t} \rangle = \langle \partial_\rho \mid \partial_t \rangle$, let D_ρ, D_t be covariant derivs.

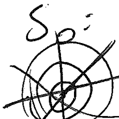
Then $\partial_\rho \langle \partial_\rho \mid \partial_t \rangle = \langle \underbrace{D_\rho \partial_\rho}_{0 \text{ geodesic}} \mid \partial_t \rangle + \langle \partial_\rho \mid D_\rho \partial_t \rangle$

$$\tau(\partial_\rho, \partial_t) = D_\rho \partial_t - D_t \partial_\rho - \underbrace{[\partial_\rho, \partial_t]}_{0} = 0 \Rightarrow D_\rho \partial_t = D_t \partial_\rho, \text{ so}$$

$$\partial_\rho \langle \partial_\rho \mid \partial_t \rangle = \langle \partial_\rho \mid D_t \partial_\rho \rangle = \tfrac{1}{2} \partial_t \langle \partial_\rho \mid \partial_\rho \rangle = \tfrac{1}{2} \partial_t \|\partial_\rho\|^2 = 0.$$

So $\langle \partial_\rho \mid \partial_t \rangle$ is constant wrt ρ. If $\rho = 0$, then $\partial_t = 0$, so $\langle \partial_\rho \mid \partial_t \rangle = 0 \ \forall \rho, t$. $\Rightarrow \langle \theta_r'(0) \mid \gamma_v'(r) \rangle = 0$ //.

So: every geodesic through p is orthogonal to $\exp_p(S_r)$ $\forall$ suff small r.



- like polar coords. Use this to calc lengths.

Lemma: Let $0 < a < b < r$. If $w(t) = \exp_x(\rho(t), \theta(t))$ is a path from $\exp S_a$ to $\exp S_b$, then $\ell(w) \geq b - a$, with equality iff θ is const, ρ is monotone — i.e., w parameterizes a geodesic.

Pf: Let $f(\rho, t) = \exp_x \rho\theta(t)$. Then $\langle \frac{\partial f}{\partial \rho} \mid \frac{\partial f}{\partial t} \rangle = 0$ and $\|\frac{\partial f}{\partial \rho}\| = 1$. Since $w(t) = f(\rho(t), t)$, $w'(t) = \rho'(t)\frac{\partial f}{\partial \rho} + \frac{\partial f}{\partial t}$

$$\|w'(t)\| = \sqrt{|\rho'(t)|^2 + \left\|\frac{\partial f}{\partial t}\right\|^2} \geq |\rho'(t)|.$$

and $l(w) \ge \int_0^1 |\rho'(t)|\,dt \ge \rho(1) - \rho(0) = b-a$.
If $l(w) = b-a$, these are sharp $\Rightarrow \rho'(t) \ge 0\ \forall t$, $\|\frac{\partial \theta}{\partial t}\| = 0 \Rightarrow$ θ constant, ρ monotone. //

Pf of Thm: Suppose $\exp_x$ is a diffeo on $B_{r_0}(0)$. Let $y \in \exp_x B_{r_0}(0)$, let w connect x to y. Then w contains a segment from $\exp S_\varepsilon$ to $\exp S_r$, so $l(w) \ge r - \varepsilon$. $\Rightarrow l(w) \ge r$. $\Rightarrow d(x,y) \ge r$.

Suppose $l(w) = r$ — ~~then the~~ then

the segment from $\exp S_\varepsilon$ to $\exp S_r$ ~~is~~ is a segment of a geodesic $\forall \varepsilon \Rightarrow w$ is a ~~g~~ reparameterization of a geodesic.

Conversely, if $d(x,z) < r_0$, then $z \in \exp_x(B_{r_0}(0)) = W$. //

Def: For $x \in M$, the <u>injectivity radius</u> of x is
$$\mathrm{injrad}(x) = \sup\{r : \exp_x \text{ is a diffeo on } B_r(0)\}.$$
If $d(x,y) < \mathrm{injrad}(x)$, then $\exists!$ geod of length $d(x,y)$ from x to y, which is the unique shortest path from x to y.

~~Further Unif NN Thm implies that $\forall p \in M$, $\exists \varepsilon > 0$, $W \ni p$ s.t. $\mathrm{injrad}(x) > \varepsilon\ \forall x \in W$.~~ Conversely?
~~Remark~~ Prop: Let $\gamma : [0,L] \to M$ be a piecewise-smooth, length-minimizing path with constant speed. Then γ is a geodesic.

Pf: Geodesic is a local condition, so it suffices to check locally. Let $t \in [0,L]$. By UNN thm, $\exists\, W \ni \gamma(t)$, $\varepsilon > 0$ s.t. $\mathrm{injrad}(x) > \varepsilon$ $\forall x \in W$. By compactness, $\exists\, \delta > 0$ s.t. $\mathrm{injrad}(\gamma(t)) > \delta\ \forall t \in [0,L]$.

Let $t \in (0,L)$. Let $r = \min\{t, L-t, \frac{\delta}{3}\}$. Consider $\gamma(t-r)$ and $\gamma(t+r)$. We have $d(\gamma(t-r), \gamma(t+r)) \le 2r < \delta$, so $\exists!$ length-minimizing path ~~geod~~ from $\gamma(t-r)$ to $\gamma(t+r)$ — $\gamma|_{[t-r,t+r]}$ must be this path, so $\gamma|_{[t-r,t+r]}$ is a geodesic.
$\Rightarrow D_t \gamma' = 0$ on $[t-r, t+r] \Rightarrow D_t \gamma' = 0\ \forall t$. //
So we can find geods by looking for length-minimizing paths —
Next time: When does a length-minimizing path from p to q exist?

2025-03-03

Last time: Geodesics and length-minimizers.

- UNN theorem: $\forall p \in M$, $\exists$ $W \ni p$, $\varepsilon > 0$ s.t.

① $\forall x, y \in W$ $\exists!$ geod of length $< \varepsilon$ from x to y

② γ depends smoothly on x and y

③ $\forall x \in W$, $\exp_x$ sends $B_\varepsilon(0)$ diffeo to M, $\exp_x B_\varepsilon(0) \supset W$.

Def: the injectivity radius $\mathrm{injrad}(x)$ of M at x is ~~the~~

~~Thm~~ $\mathrm{injrad}(x) = \sup\{r \mid \exp_x \text{ is a diffeo on } B_r(0)\}$.

Thm (geodesics shorter than inj rad are length-minimizers): Let $x \in M$, $r_0 = \mathrm{injrad}(x)$. ~~① Any geod from x of length $< r_0$ is length-minimizing. ② if $y \in \exp_x B_{r_0}(0)$, then x is connected~~ to ~~y by a unique~~ length-minimizing curve.

~~① If $v \in T_xM$ [illegible] If $y \in \exp_x B_{r_0}(0)$, then y is conne~~

② If $y = \exp_x(v)$, $v \in T_xM$, $\|v\| = r < r_0$, then $\exists$ unique length-minimizing curve from x to y, which is a geod of length r.

Thus: ① any geodesic of length $< r_0$ from x is length-minimizing.

② if $d(x,y) < r_0$, then $y \in \exp_x B_{r_0}(0)$

③ if $r < r_0$, then $\{y \in M \mid d(x,y) < r\} = \exp_x B_r(0)$.

Today: ① length-minimizers are geodesics

② Hopf-Rinow: if M is complete, then any x, y are connected by a length-minimizing path.

Prop: Let $\gamma : [0,1] \to M$ be a piecewise-smooth, length-minimizing path with const. speed. Then γ is a geodesic.

Pf: Rescale so that $\gamma : [0, L] \to M$ and $\|\gamma'\| = 1$.

Claim: $D_t \gamma' = 0$ $\forall t$. By UNN ~~above~~, $\forall t \in [0, L]$, $\exists$ $W \ni \gamma(t)$, $\varepsilon > 0$ s.t. $\mathrm{injrad}(w) \geq \varepsilon$ $\forall w \in W$. By compactness, $\exists \varepsilon > 0$ s.t. $\mathrm{injrad}(\gamma(t)) \geq \varepsilon$ $\forall t$. Let $t \in (0, L)$, let $r = \min\{t, L-t, \frac{\varepsilon}{3}\}$. ~~Consider $\gamma(t-r), \gamma(t+r)$~~ Then $d(\gamma(t-r), \gamma(t+r)) < \frac{2\varepsilon}{3}$, so $\exists!$ length-min curve from $\gamma(t-r)$ to $\gamma(t+r)$ — $\gamma|_{[t-r, t+r]}$ parameterizes this curve with const speed $\Rightarrow$ γ is a geod near t $\Rightarrow$ $D_t \gamma'(t) = 0$ //.

So any shortest path is a geod — when do shortest paths exist? When M is compact (Arzela-Ascoli), when M is ~~[illegible]~~ a proper metric space (~~[illegible]~~ all closed balls are compact). In general?

~~Thm~~ Def: M is geodesically complete if $\forall p \in M$, $\forall v \in T_pM$, $\exists$ a geod γ, defined on all of $\mathbb{R}$, s.t. $\gamma(0) = p$, $\gamma'(0) = v$.

Equiv, $\forall p$, $\exp_p$ is defined on all of T_pM.

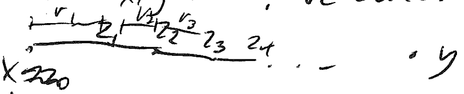
Thm (Hopf-Rinow): If M is connected and geodesically complete, then $\forall x,y \in M$ $\exists$ a length-minimizing geodesic from x to y. ~~Ex: converse~~ Ex: $\mathbb{R}^2 \setminus \{0\}$.

Lemma: let $x,y \in M$, $x \neq y$, let $L = d(x,y)$, $0 < r < \min(L, \text{injrad}(x))$. Then $\exists z \in M$ s.t. $d(x,z) = r$ and $d(z,y) = L - r$.

Pf: Let $S = \exp_x(S_r)$. ~~This~~ where $S_r = \{v \in T_xM \mid \|v\| = r\}$. This is compact, so $\exists z \in S$ s.t. $d(z,y) = \min_{s\in S} d(s,y)$. Then $d(z,x) = r$. By triangle ineq, $d(x,y) \leq \min_{s\in S}(d(x,s) + d(s,y))$. But every path from x to y crosses S, so $d(x,y) \geq \min_{s\in S}(d(x,s) + d(s,y)) \Rightarrow$

$$L = \min_{s\in S} d(x,s) + d(s,y) = \min_{s\in S}(r + d(s,y)) = r + d(z,y) \Rightarrow d(z,y) = L - r \;//$$

~~So we~~ let $x,y \in M$. We can construct a sequence of steps:



~~s.t. $d(x,y) = d(x,z_i) + d(z_i,y)$ $\forall i$ by induction~~

then $d(x, z_i) \leq r_1 + \dots + r_i$

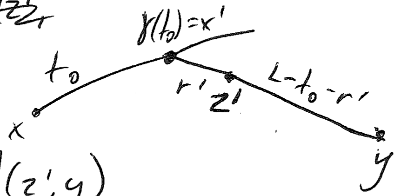
$d(z_i, y) = d(x,y) - r_1 - \dots - r_i$, so it by triangle, $d(x,z_i) = r_1 + \dots + r_i$

so this path is a length-minimizing geodesic. Only thing left is to show that this eventually hits y. — for that, we need geod completeness.

Pf: let $x,y \in M$ ~~let~~ let $L = d(x,y)$. If $L < \text{injrad}(x)$ we're done. Otherwise, let $0 < r < \text{injrad}(x)$, let z s.t. $d(x,z) = r$, $d(y,z) = L - r$. Then $\exists!$ unit speed geodesic γ s.t. $\gamma(0) = x$, $\gamma(r) = z$. Claim: $\forall t \in [0,L]$, $d(\gamma(t), y) = L - t$. — Then $d(\gamma(L), y) = 0 \Rightarrow \gamma(L) = y$.

Let $t_0 = \sup\{t \in [0,L] \mid d(\gamma(t), y) = L - t\}$. (Note: $d(\gamma(t),y) = L - t$ $\forall t \in [0,t_0]$ by Δ-ineq?) If $t_0 = L$, we're done. Otherwise, by ~~the lemma, $\exists z \in M$~~ let $x' = \gamma(t_0)$. Let $0 < r' < \text{injrad}(x')$ and let $z' \in M$ s.t.

$$d(x', z') = r', \quad d(z', y) = L - t_0 - r'.$$



By Δ-ineq, $d(x, z') \geq d(x,y) - d(z',y)$

$= L - (L - t_0 - r') \geq t_0 + r'$

So the path from x to x' to z' is length-minimizing $\Rightarrow$ geodesic with initial velocity $\gamma'(0) \Rightarrow z' = \gamma(t_0 + r')$, contradicting the maximality of t_0. //

Consequences:

Cor: If M is geodesically complete, and $\overline{B_r}(x)=\{y\in M \mid d(x,y)<r\}$, then $\overline{B_r}(x) = \exp_x(\overline{B_r}(0))$. In particular, $\overline{B_r}(x)$ is compact. (image of a compact set)

Cor: M complete (as a metric space) $\Longleftrightarrow$ M is geodesically complete.

Pf: Exercise

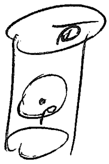
~~Cor: If M is compact, then any~~ connected

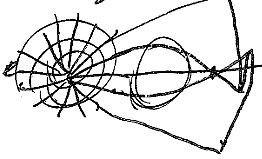
~~Cor: If M is geodesically complete, then $\exp_x$ is surjective.~~

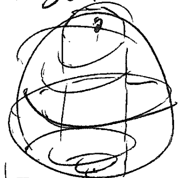
When ~~does~~ is a geodesic minimal/not minimal?

— As long as $l(\gamma) < \mathrm{injrad}(0)$, γ is minimizing.

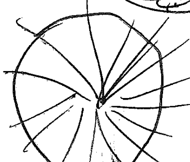
When $l(\gamma) \geq$ injrad, ~~start getting~~ can construct examples.

Ex:  cylinder, p on one side, ~~when $r \geq 2\pi p$, the ball~~ the exp isn't injective on $B_{\pi r}(0)$. Geodesics fail to be injective because there's a geodesic loop.

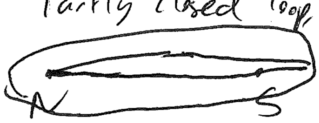
Not always the case: Imagine  — exp has a singularity, three geods from x to y, one of them is not minimal.

Ex:  S^2

$\exp_p$ has singularity on $B_0(\pi)$



Geodesic past this singularity isn't minimal.
Partly closed loop, but even if we cut, there's a shorter path.



To explore this, Calc of Var.. Questions?

~~Overflow:~~ $\Omega_{p,q}$, Overflow: