Liouville–Arnold connection is torsion-free

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1 Torsion

Let $X \to B$ be a Kovalev–Lefschetz pencil. Then one has an isomorphism $T_b B \cong H^+(X_b, \mathbb{R}) \subset H^2(X_b, \mathbb{R})$ given by $v \mapsto [\iota_{\widetilde{v}}\rho|_{X_b}]$ where $\widetilde{\cdot}$ denotes the horizontal lift. This restriction is closed by the McLean's characterization.

Consider the vector bundle $E \to X$ given by $E_b = H^2(X_b, \mathbb{R})$. It is equipped with the fiberwise Poincaré pairing $([\alpha], [\beta]) = \int_{X_b} \alpha \wedge \beta$. The above construction realizes the tangent bundle TB as a subbundle $E^+ \subset E$. The restriction of the Poincaré pairing onto $TB \subset E$ is positive definite. Let $\widehat{T}B \subset E$ be the orthogonal complement of TB w. r. t. the Poincaré pairing. It is negative definite on the bundle $\widehat{T}B$. From now on, one shall write $u^\S \in \Gamma(E)$ for a vector field $u \in \Gamma(TB)$ when one considers it as a section of the bundle E. We shall refer to the bundle $\widehat{T}B$ as to the **antitangent bundle**.

Consider the Gauß–Manin connection $\nabla^{GM} \colon TB \otimes E \to E$ or rather its restriction onto $TB \subset E$. It is given by the direct sum of two operators: $\nabla^{LA} \oplus Q \colon TB \otimes TB \to TB \oplus \widehat{T}B$.

Proposition 1. *The operator ∇^{LA} is a connection.*

Proof. Denote the orthogonal projection $E \to TB$ by π^+ (or simply π), and $E \to \widehat{T}B$ by π^-. One has

$$\nabla^{LA}_{fu} v = \pi\left(\nabla^{GM}_{fu} v\right) = \pi\left(f \nabla^{GM}_u v\right) = f\pi\left(\nabla^{GM}_u v\right) = f\nabla^{LA}_u v.$$

What about the Leibniz rule, one has

$$\nabla^{LA}_u (fv) = \pi\left(\nabla^{GM}_u (fv)\right) = \pi\left(f\nabla^{GM}_u v + (\mathrm{Lie}_u f)\, v\right) = f\nabla^{GM}_u v + (\mathrm{Lie}_u f)\, \pi(v) = f\nabla^{GM}_u v + (\mathrm{Lie}_u f)\, v$$

since $v \in \Gamma(TB) \subset \Gamma(E)$. □

Definition 1. The above operator ∇^{LA} is called the **Liouville–Arnold connection**.

Lemma 1. *Let k be an integer, $p \colon Y \to A$ a fibration, and $F \to A$ be the k-th cohomology bundle, i. e. the bundle with fibers $F_a = H^k(Y_a, \mathbb{R})$. Let $w \in \Gamma(TA)$ be a vector field, and $s \in \Gamma(F)$ be a section s. t. there exists a form $\alpha \in \Omega^k(Y)$ s. t. $d\alpha|_{Y_a} = 0$ for any $a \in A$, and $[\alpha|_{Y_a}] = s_a$. Then the Gauß–Manin connection $\nabla^{GM} \colon TA \otimes F \to F$ is given by*

$$\left(\nabla^{GM}_w s\right)_a = \left[\mathrm{Lie}_{\widetilde{w}}\, \alpha|_{Y_a}\right],$$

when $\widetilde{w}$ is any lift of the vector field w.

Proof. Let us show that the right-hand side is independent on choises made. To show that it is independent on the lift of the field w, it is enough to show that it vanishes when $\widetilde{w}$ is a vertical field. But the flow of a vertical field preserves the fibers Y_a, and the cohomology classes as well. In such a case the cohomology class of a derivative is the derivative of a cohomology class, and since the flow acts by isotopies, the derivative vanishes.

A form α restricts as identical zero to all the fibers iff it can be expressed as a linear combination of forms pulled back from the base A with coefficients in functions on Y:

$$\alpha = \sum_i \psi_i p^*(\alpha_i),\ \psi_i \in C^\infty(Y),\ \alpha_i \in \Omega^k(A).$$

Then one has $\operatorname{Lie}_{\widetilde{w}} \alpha = \sum_i (\operatorname{Lie}_{\widetilde{w}} \psi_i)\, p^*(\alpha_i) + \psi_i \operatorname{Lie}_{\widetilde{w}} p^*(\alpha_i) = \sum_i (\operatorname{Lie}_{\widetilde{w}} \psi_i)\, p^*(\alpha_i) + \psi_i p^*(\operatorname{Lie}_w \alpha_i)$. Hence $\operatorname{Lie}_{\widetilde{w}} \alpha|_{Y_a} = 0$ for any fiber Y_a.

Now let α restricts as an exact form onto each fiber Y_a, i. e. for any $a \in A$ there exists a form $\beta_a \in \Omega^{k-1}(Y_a)$ s. t. $\alpha|_{Y_a} = d\beta_a$. Consider any form $\beta \in \Omega^{k-1}(Y)$ s. t. for any $a \in A$ one has $\beta|_{Y_a} = \beta_a$ (say by picking up a horizontal subbundle and defining the form β as β_a along the fiber Y_a and 0 along the horizontal subspaces). Then one has $[\operatorname{Lie}_{\widetilde{w}} \alpha|_{Y_a}] = [\operatorname{Lie}_{\widetilde{w}} \alpha - d\beta|_{Y_a}]$. However, the form $\alpha - d\beta$ vanishes along the fibers, hence its derivative also vanishes.

Now one can check that the right-hand side defines a connection. Indeed, for any function $\varphi \in C^\infty(A)$ one has $\left[\operatorname{Lie}_{\varphi\widetilde{w}} \alpha|_{Y_a}\right] = [\varphi(a)\iota_{\widetilde{w}} d\alpha|_{Y_a}] = \varphi(a)\,[\iota_{\widetilde{w}}\alpha|_{Y_a}]$ and $[\operatorname{Lie}_{\widetilde{w}}(\varphi\alpha)|_{Y_a}] = [\operatorname{Lie}_{\widetilde{w}}(\varphi)\alpha|_{Y_a} + \varphi \operatorname{Lie}_{\widetilde{w}} \alpha|_{Y_a}] = (\operatorname{Lie}_w \varphi)\,[\alpha|_{Y_a}] + \varphi(a)\,[\operatorname{Lie}_{\widetilde{w}} \alpha|_{Y_a}]$.

Since it is a connection, it must be the Gauß–Manin connection by the naturality of the construction. □

Lemma 2. *One has* $\iota_{[u,v]} = [\operatorname{Lie}_u, \iota_v]$.

Proof. Since both sides are the derivations of the de Rham algebra, one suffies to check this on generators, i. e. functions end exact forms. On functions both operator vanish, hence are trivially equal. On the exact form df one has $\iota_{[u,v]} df = \operatorname{Lie}_{[u,v]} f = \operatorname{Lie}_u \operatorname{Lie}_v f - \operatorname{Lie}_v \operatorname{Lie}_u f = \operatorname{Lie}_u \iota_v df - (\iota_v d + d\iota_v)\,\iota_u df = \operatorname{Lie}_u \iota_v df - \iota_v d\iota_u df = \operatorname{Lie}_u \iota_v df - \iota_v\,(d\iota_u + \iota_u d)\,df = [\operatorname{Lie}_u, \iota_v]\,df$. □

Proposition 2. *The torsion of the Liouville–Arnold connection vanishes.*

Proof. Let us calculate.

One has $\operatorname{Tors}^{\nabla^{LA}}(u,v) = \nabla^{LA}_u v - \nabla^{LA}_v u - [u,v]$. By Lemma 1, one has $\left(\nabla^{GM}_u v\right)^\S_b = \left[\operatorname{Lie}_{\widetilde{u}} v^\S|_{X_b}\right] = [\operatorname{Lie}_{\widetilde{u}} \iota_{\widetilde{v}}\rho|_{X_b}]$, hence for the Liouville–Arnold connection one has $\left(\nabla^{LA}_u v\right)^\S_b = \pi\,[\operatorname{Lie}_{\widetilde{u}} \iota_{\widetilde{v}}\rho|_{X_b}]$.

Let $\Theta(u,v) = \widetilde{[u,v]} - [\widetilde{u},\widetilde{v}]$ be the curvature of the horizontal distribution. It is a form with coefficients in vertical fields. One has $[u,v]^\S_b = \left[\iota_{\widetilde{[u,v]}}\rho|_{X_b}\right] = \left[\iota_{\Theta(u,v)}\rho|_{X_b}\right] + \left[\iota_{[\widetilde{u},\widetilde{v}]}\rho|_{X_b}\right]$. Since $\rho|_{X_b} \equiv 0$, and the field $\Theta(u,v)$ is vertical, i. e. tangent to the fiber X_b, the first summand vanishes. Therefore by the Lemma 2, one can write $[u,v]^\S_b = \left[\iota_{[\widetilde{u},\widetilde{v}]}\rho|_{X_b}\right] = [\operatorname{Lie}_{\widetilde{u}} \iota_{\widetilde{v}}\rho|_{X_b} - \iota_{\widetilde{v}} \operatorname{Lie}_{\widetilde{u}} \rho|_{X_b}]$.

Finally, one can write $\operatorname{Tors}^{\nabla^{LA}}(u,v)^\S_b = \left(\nabla^{LA}_u v\right)^\S_b - \left(\nabla^{LA}_v u\right)^\S_b - [u,v]^\S_b = \pi\,[\operatorname{Lie}_{\widetilde{u}} \iota_{\widetilde{v}}\rho|_{X_b}] - \pi\,[\operatorname{Lie}_{\widetilde{v}} \iota_{\widetilde{u}}\rho|_{X_b}] - [\operatorname{Lie}_{\widetilde{u}} \iota_{\widetilde{v}}\rho|_{X_b} - \iota_{\widetilde{v}} \operatorname{Lie}_{\widetilde{u}} \rho|_{X_b}]$. Since $[d\eta] = 0$, one has $[\operatorname{Lie}_{\widetilde{v}} \iota_{\widetilde{u}}\rho|_{X_b}] = [\iota_{\widetilde{v}} d\iota_{\widetilde{u}}\rho|_{X_b}]$. Since $d\rho = 0$, one also has $\iota_{\widetilde{v}} \operatorname{Lie}_{\widetilde{u}} \rho = \iota_{\widetilde{v}} d\iota_{\widetilde{u}}\rho$. Hence the formula can be simplified to

$$\operatorname{Tors}^{\nabla^{LA}}(u,v)^\S_b = (\pi - 1)\,[\operatorname{Lie}_{\widetilde{u}} \iota_{\widetilde{v}}\rho|_{X_b}] + (1-\pi)\,[\iota_{\widetilde{v}} d\iota_{\widetilde{u}}\rho|_{X_b}] = \pi^-\,[\iota_{\widetilde{v}} d\iota_{\widetilde{u}}\rho|_{X_b} - \operatorname{Lie}_{\widetilde{u}} \iota_{\widetilde{v}}\rho|_{X_b}]\,.$$

Hence the no matter what are the vector fields $u, v \in \Gamma(TB)$, the torsion $\operatorname{Tors}^{\nabla^{LA}}(u,v)$ of the Liouville–Arnold connection evaluated at these fields considered as a section of the cohomology bundle E falls into the subbundle $\widehat{T}B$, which is the range of the projection π^-. However, the value of the torsion tensor is always a vector field. Therefore $\operatorname{Tors}^{\nabla^{LA}}(u,v) \in TB \cap \widehat{T}B \subset E$. These two subbundles are perpendicular to each other. Hence $\operatorname{Tors}^{\nabla^{LA}} = 0$. □

2 Curavture

The Gauß–Manin connection is flat. This allows to write an expression for the curvature of the Liouville–Arnold connection.

Let us remind that the Gauß–Manin connection restricted to the subbundle $TB \subset E$ decomposes as $\nabla^{GM} = \nabla^{LA} \oplus Q \colon TB \otimes TB \to TB \oplus E$. One can also restrict it to the subbundle $\widehat{T}B$, so that it would decompose as $\nabla^{GM} = \widehat{Q} \oplus \widehat{\nabla}^{LA} \colon TB \otimes \widehat{T}B \to TB \oplus \widehat{T}B$. Note that one has $Q_u(fv) = \nabla^{GM}_u(fv) - \nabla^{LA}_u(fv) = f\nabla^{GM}_u v + (\mathrm{Lie}_u f)\, v - f\nabla^{LA}_u v - (\mathrm{Lie}_u f)\, v = fQ_u(v)$, hence the map $Q \colon TB \otimes TB \to \widehat{T}B$ is a linear operator. Similarly the operator $\widehat{\nabla}^{LA}$ is a connection in the bundle $\widehat{T}B$, and $\widehat{Q} \colon TB \otimes \widehat{T}B \to TB$ is a linear operator. By abuse of notation we shall refer to the connection $\widehat{\nabla}^{LA}$ in the antitangent bundle as to the Liouville–Arnold connection. We shall also call the tensors Q and $\widehat{Q}$ the **inclination** and **anti-inclination tensors**, respectively.

Proposition 3. *The curvature of the Liouville–Arnold connection is the commutant of the inclinations with opposite sign.*

Proof. Let $u, v, w \in \Gamma(TB)$ be three vector fields. One has

$$
\begin{aligned}
0 = {}& \\
&= \nabla^{GM}_u \left(\nabla^{GM}_v w^{\S}\right) - \nabla^{GM}_v \left(\nabla^{GM}_u w^{\S}\right) - \nabla^{LA}_{[u,v]} w^{\S} = \\
&= \nabla^{GM}_u \left(\left(\nabla^{LA}_v w\right)^{\S} + Q_v(w^{\S})\right) - \nabla^{GM}_v \left(\left(\nabla^{LA}_u w\right)^{\S} + Q_u(w^{\S})\right) - \left(\nabla^{LA}_{[u,w]} w\right)^{\S} - Q_{[u,v]}(w^{\S}) = \\
&= \left(\nabla^{LA}_u \left(\nabla^{LA}_v w\right)\right)^{\S} + Q_u \left(\left(\nabla^{LA}_v w\right)^{\S}\right) + \widehat{Q}_u \left(Q_v(w^{\S})\right) + \widehat{\nabla}^{LA}_u \left(Q_v(w^{\S})\right) - \\
&\quad - \left(\nabla^{LA}_v \left(\nabla^{LA}_u w\right)\right)^{\S} - Q_v \left(\left(\nabla^{LA}_u w\right)^{\S}\right) - \widehat{Q}_v \left(Q_u(w^{\S})\right) - \widehat{\nabla}^{LA}_v \left(Q_u(w^{\S})\right) - \\
&\quad - \left(\nabla^{LA}_{[u,w]} w\right)^{\S} - Q_{[u,v]}(w^{\S}) = \\
&= \left(\Theta^{\nabla^{LA}}_{u,v} w\right)^{\S} + \widehat{Q}_u \left(Q_v(w^{\S})\right) - \widehat{Q}_v \left(Q_u(w^{\S})\right) + \\
&\quad + Q_u \left(\left(\nabla^{LA}_v w\right)^{\S}\right) + \widehat{\nabla}^{LA}_u \left(Q_v(w^{\S})\right) - Q_v \left(\left(\nabla^{LA}_u w\right)^{\S}\right) - \widehat{\nabla}^{LA}_v \left(Q_u(w^{\S})\right) - Q_{[u,v]}(w^{\S}).
\end{aligned}
$$

The part of this sum lying in the subbundle $TB \subset E$ is $\left(\Theta^{\nabla^{LA}}_{u,v} w\right)^{\S} + \widehat{Q}_u \left(Q_v(w^{\S})\right) - \widehat{Q}_v \left(Q_u(w^{\S})\right)$, whereas the part lying in the antitangent bundle $\widehat{T}B$ is more complicated and reads $Q_u \left(\left(\nabla^{LA}_v w\right)^{\S}\right) + \widehat{\nabla}^{LA}_u \left(Q_v(w^{\S})\right) - Q_v \left(\left(\nabla^{LA}_u w\right)^{\S}\right) - \widehat{\nabla}^{LA}_v \left(Q_u(w^{\S})\right) - Q_{[u,v]}(w^{\S})$. Since the total sum vanishes, both of these summands also vanish. Slightly abusing the notation, one hence may write

$$\Theta_{u,v} = -\left[Q_u, Q_v\right].$$

The proof also gives an explicit first-order expression for $Q_{[u,v]}(w^{\S})$. ∎