Sections of Lagrangian fibrations on hyperkähler manifolds and degenerate twistorial deformation

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Abstract

For hyperkähler manifolds with Lagrangian fibrations we show that existence of a deformation with a section is a condition on topology of the fibration. We establish that this condition is satisfied for elliptic K3 surfaces.

Contents

1 Complex structures via symplectic forms

Let V be an $\mathbb{R}$-vector space of dimension $4n$ and $\Upsilon \in \Lambda^2 V^* \otimes \mathbb{C}$ be a complex-valued 2-form on it. Suppose that Υ is non-degenerate, i. e. $\forall u \in V \exists v \colon \Upsilon(u,v) \neq 0$. Then $\ker \Upsilon \subset V \otimes \mathbb{C}$ is a complex subspace containing no real vectors, i. e. $\ker \Upsilon \cap \overline{\ker \Upsilon} = 0$. In particular, $\dim \ker \Upsilon \leq 2n$. Provided that $\dim \ker \Upsilon = 2n$, it gives rise to a complex structure operator I_Υ on the vector space V such that $I_\Upsilon|_{\ker \Upsilon} = -\sqrt{-1}\,\mathrm{id}$ and $I_\Upsilon|_{\ker \overline{\Upsilon}} = \sqrt{-1}\,\mathrm{id}$.

Definition 1. A *complex symplectic space* is a pair (V, Υ), where $\Upsilon \in \Lambda^2 V^* \otimes \mathbb{C}$, such that $\ker \Upsilon \cap V = \{0\}$ and $\ker \Upsilon$ has maximal possible rank.

Proposition 1.1. *A nondegenerate complex-valued* 2*-form* Υ *is complex symplectic if and only if any real vector is the real part of a vector from* $\ker \Upsilon$*.*

Proof. Obvious. □

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Proposition 1.2. *One has $\Upsilon(I_\Upsilon u, v) = \sqrt{-1}\Upsilon(u,v)$. In particular, the complex structure I_Υ preserves subspaces which are orthogonal w. r. t. Υ.*

Proof. By definition of the complex structure I_Υ, $\forall u \in V$ one has $u + \sqrt{-1}I_\Upsilon u \in \ker\Upsilon$. Thence $\Upsilon(I_\Upsilon u, v) = \frac{\Upsilon(\sqrt{-1}I_\Upsilon u, v)}{\sqrt{-1}} = \frac{\Upsilon(u+\sqrt{-1}I_\Upsilon u, v) - \Upsilon(u,v)}{\sqrt{-1}} = -\frac{\Upsilon(u,v)}{\sqrt{-1}} = \sqrt{-1}\Upsilon(u,v)$. □

Proposition 1.3 (N. Hitchin [**Hi**])**.** *Any Lagrangian subspace w. r. t. a complex symplectic form is complex w. r. t. the corresponding complex structure.*

Proof. By definition, a Lagrangian subspace is the orthogonal of itself. Then it is invariant under I_Υ due to Proposition 1.2. □

In particular, quotient by a Lagrangian vector subspace inherits a complex structure. Note that this complex structure on the quotient is not in general determined by any complex symplectic form: indeed, the quotient may have odd complex dimension.

Proposition 1.4 (Darboux type theorem)**.** *Complex symplectic space possesses a basis in which the complex symplectic form is given by a block diagonal matrix with equal blocks Q on the diagonal, where Q stands for a 4×4-block given by*

$$Q = \begin{pmatrix} 0 & 0 & 1 & \sqrt{-1} \\ 0 & 0 & \sqrt{-1} & -1 \\ -1 & -\sqrt{-1} & 0 & 0 \\ -\sqrt{-1} & 1 & 0 & 0 \end{pmatrix}.$$

Proof. The Proposition is proved by running an analogue of the Gram–Schmidt process. For the first two vectors in the basis we may choose an arbitrary nonzero vector u_1 and its image under complex structure, $I_\Upsilon u_1$. Orthogonals thereof coincide due to the previous Proposition. If one picks a vector u_2 outside this orthogonal, then, by outstretching it and adding to it a multiple of the vector $I_\Upsilon u_2$ in case of necessity, we can make u_2 such that

$$\Upsilon(u_1, u_2) = 1.$$

One can deduce from Proposition 1.2 and the above relation that the restriction of the form Υ onto the four-dimensional subspace $U \subset V$ spanned by $\{u_1, I_\Upsilon u_1, u_2, I_\Upsilon u_2\}$ is given by the matrix Q in these coordinates.

Now we can proceed, excercising the same procedure in the orthogonal to U, because $U \cap U^\perp = \{0\}$. □

Proposition 1.5. *The following are equivalent:*

1. $\dim\ker\Upsilon = 2n$,
2. $(\Upsilon \wedge \overline{\Upsilon})^n$ *is nonzero and* $\Upsilon^{n+1} = 0$.

Proof. Suppose that $\dim\ker\Upsilon = 2n$. Then for any decomposable polyvector $a \in \Lambda^{2n+2}V \otimes \mathbb{C}$ the corresponding subspace in V needs to intersect $\ker\Upsilon$, so $\iota_a\Upsilon^{n+1}$ vanishes. Hence $\Upsilon^{n+1} = 0$. An easy direct calculation shows that, in notation of the Proposition 1.3, $(Q \wedge \overline{Q})(u_1, v_1, u_2, v_2) = 4$, so the top power of $\Upsilon \wedge \overline{\Upsilon}$ cannot be zero.

Suppose that $\Upsilon^{n+1} = 0$. If the top power of a skew-symmetric form on some vector space is zero, then this form has a nontrivial kernel. That's why the form Υ has nontrivial kernel when restricted to any $(2n+2)$-plane inside $V \otimes \mathbb{C}$ (and, moreover, any $(2n+2k)$-plane for any $k > 0$). Restriction of a symplectic form onto a complement of its kernel is non-degenerate, so $\dim\ker\Upsilon \geqslant 2n$. If $(\Upsilon \wedge \overline{\Upsilon})^n$, it cannot be greater, because in this case $\ker\Upsilon$ needs to intersect $\ker\overline{\Upsilon}$, thus giving a real vector in $\ker\Upsilon$, substitution of which would vanish $(\Upsilon \wedge \overline{\Upsilon})^n$. □

Proposition 1.6. *Let V be a vector space, Υ a complex symplectic form on it, L a Lagrangian subspace and $\sigma\colon V/L \to V$ a real section (not necessarily complex linear). Define the form $\Upsilon_* \in \Lambda^2(V/L)^* \otimes \mathbb{C}$ by the rule $\Upsilon_*(u_1, u_2) = \Upsilon(\sigma(u_1), \sigma(u_2))$ (i. e. as restriction of Υ onto the subspace $\sigma(V/L)$ after the identification $\pi|_{\sigma(V/L)}\colon \sigma(V/L) \to V/L$). Then this form Υ_* has type $(2,0)+(1,1)$ in the complex structure on V/L coming from V.*

Proof. Suppose that $\sigma = \sigma_0$ is complex linear. Then the form Υ_* is of type $(2,0)$, since the type is preserved by complex linear maps.

Now let $\sigma = \sigma_0 + \tau$, where σ_0 is a complex linear section and $\tau\colon V/L \to L$ is some real perturbation. Then for $u, v \in V/L$ one has $\Upsilon_*(u,v) = \Upsilon(\sigma(u), \sigma(v)) = \Upsilon(\sigma_0(u), \sigma_0(v)) + \Upsilon(\sigma_0(u), \tau(v)) + \Upsilon(\tau(u), \sigma_0(v)) + \Upsilon(\tau(u), \tau(v))$. Since one has $\tau(u), \tau(v) \in L$ and L is a Lagrangian subspace w. r. t. Υ, the last term disappears. The first term is a $(2,0)$ form, since σ_0 is a complex linear section. The term $\Upsilon(\sigma_0(u), \tau(v)) + \Upsilon(\tau(u), \sigma_0(v))$ vanishes if u, v are both of type $(0,1)$, since both $\sigma_0(u)$ and $\sigma_0(v)$ are also of type $(0,1)$ in this case, and therefore annihilate the form Υ. Hence the $(0,2)$ part of the form Υ_* vanishes, so it is of type $(2,0)+(1,1)$. □

Fiberwise application of the above construction allows one to obtain an almost complex structure on a $4n$-dimensional manifold X from a non-degenerate complex-valued 2-form Ω on it such that $(\Omega\wedge\overline{\Omega})^n$ is nowhere zero and $\Omega^{n+1} = 0$.

Definition 2. An *almost complex symplectic form* on a manifold X of real dimension $4n$ is a form $\Omega \in \Gamma(\Lambda^2 T^*X \otimes \mathbb{C})$ such that the top degree form $(\Omega \wedge \overline{\Omega})^n$ is nonzero and $\Omega^{n+1} = 0$. A *complex symplectic form* is a closed almost complex symplectic form.

We shall denote the almost complex structure obtained from the form Ω by I_Ω.

Proposition 1.7. *For an almost complex symplectic form Ω the almost complex structure I_Ω is integrable if $d\Omega = 0$. In particular, the complex structure on a holomorphically symplectic manifold is determined by its holomorphic symplectic form alone.*

Proof. It is known since a long time ago (see e. g. formula (2.2) in [**KS**]) that $\iota_{[u,v]} = [\mathrm{Lie}_u, \iota_v]$. Indeed, both terms are de Rham algebra derivations, so it suffices to check that they coincide on functions (they both vanish on them) and exact 1-forms. That is straightforward: $\iota_{[u,v]}df = \mathrm{Lie}_{[u,v]}f = \mathrm{Lie}_u\mathrm{Lie}_v f - \mathrm{Lie}_v\mathrm{Lie}_u f = \mathrm{Lie}_u\mathrm{Lie}_v f - \iota_v d\iota_u df = \mathrm{Lie}_u\iota_v df - \iota_v\mathrm{Lie}_u df = [\mathrm{Lie}_u, \iota_v]df$.

If u and v are antiholomorphic vector fields w. r. t. the complex structure I_Ω (i. e. $\iota_u\Omega = \iota_v\Omega = 0$), we have $\iota_{[u,v]}\Omega = \mathrm{Lie}_u\iota_v\Omega - \iota_v\mathrm{Lie}_u\Omega = -\iota_v d\iota_u\Omega - \iota_v\iota_u d\Omega = \iota_u\iota_v d\Omega$. Therefore $[u,v]$ is antiholomorphic, when $d\Omega = 0$. □

Note that converse is not generally true: if Ω is a complex symplectic form and f is a nowhere zero function, then $f\Omega$ is not closed unless f is non-constant, and $I_{f\Omega} = I_\Omega$.

2 Degenerate twistorial deformation

Though this section is self-contained, it is inspired by the paper [**Ve**] of Verbitsky.

Proposition 2.1. *Suppose that $\Upsilon \in \Lambda^2 V^* \otimes \mathbb{C}$ is a complex symplectic form on a vector space V, L is a Lagrangian subspace in V and $\pi\colon V \to V/L$ the projection. Then for any form $\gamma \in \Lambda^2(V/L)^* \otimes \mathbb{C}$ of type $(2,0)+(1,1)$ the forms $\Upsilon_t = \Upsilon + t\pi^*\gamma$ are complex symplectic.*

Proof. We shall proceed in two steps: first, we prove that $\dim\ker\Upsilon_t$ is at least half of $\dim V$, and then we prove that $\dim\ker\Upsilon$ does not exceed half of $\dim V$.

Step 1. If we vary the complex symplectic form slightly by adding a multiple of the form $\pi^*\gamma$ to it, i. e. $\Upsilon_t = \Upsilon_0 + t\pi^*\gamma$, provided it stays complex symplectic, then its kernel $\ker\Upsilon_t$ would also move. Such a movement can be given (at least u. t. the first order in t) by a linear map $\zeta\colon \ker\Upsilon \to L$ s. t. for $v = v_0 \in \ker\Upsilon_0$ one has $v_t = v_0 + t\zeta(v) \in \ker\Upsilon_t$. If we succeed in constructing such a map $\zeta\colon \ker\Upsilon \to L$, then Step 1 is proven.

Indeed, the condition

$$v_0 + \zeta(v) = v_t \in \ker\Upsilon_t = \ker(\Upsilon + t\pi^*\gamma)$$

can be expanded as

$$\Upsilon(v) + t\,(\pi^*\gamma(v) + \Upsilon(\zeta(v))) + t^2\pi^*\gamma(\zeta(v)) = 0.$$

Since the form $\pi^*\gamma$ vanishes on the kernel $L = \ker\pi$, the t^2-term vanishes, as well as the constant term. Therefore the only condition on the map $\zeta\colon \ker\Upsilon \to L$ to give the variation of the kernel reads $\pi^*\gamma(v) = -\Upsilon(\zeta(v))$. Since γ is of type $(2,0)+(1,1)$ w. r. t. the complex structure given by Υ, so is $\pi^*\gamma$, and since $v \in \ker\Upsilon$, i. e. v is of $(0,1)$ type, the contraction $\pi^*\gamma(v)$ has type $(1,0)$. One can think of it as of a $(1,0)$ form on V/L. However, the form Υ is a nondegenerate $(2,0)$ form with L as a Lagrangian subspace, so it establishes an isomorphism

$\Upsilon\colon L^{1,0} \to ((V/L)^*)^{1,0}$ given by plugging in the vectors. Now the desired map ζ can be given as

$$\zeta(v) = -\Upsilon^{-1}(\iota_v(\pi^*(\gamma))) = -\Upsilon^{-1}(\iota_{\pi(v)}\gamma).$$

Step 2. If one has $\dim\ker\Upsilon_t > \frac{1}{2}\dim V$, then Υ_t has a real vector in its kernel, say w. This vector satisfies $\iota_w\Upsilon = -t\iota_w\pi^*\gamma = \pi^*\iota_{-t\pi(w)}\gamma$, in particular, $\iota_w\Upsilon$ vanishes on L. This means that w is Υ-orthogonal to L, i. e. $w \in L$ and $\iota_w\pi^*\gamma = 0$. On the other hand, one has $0 = \iota_w\Upsilon_t = \iota_w\Upsilon + t\iota_w\pi^*\gamma = \iota_w\Upsilon$, which contradicts the nondegeneracy of the form Υ. Therefore one has $\dim\ker\Upsilon_t = \frac{1}{2}\dim V$. □

It is worth mentioning that this Proposition fails in the case when the form γ has nonzero Hodge $(0,2)$-part, since the form $\pi^*\gamma(v)$ can have nonzero $(0,1)$ part in this case.

Proposition 2.2. *In the notation of the Proposition 2.1, the subspace L is Lagrangian w. r. t. all the forms Υ_t, and restrictions $I_{\Upsilon_t}|_L$ onto L of complex structures I_{Υ_t} are the same. Moreover, the complex structures I_{Υ_t} induced on the quotient V/L are also all the same.*

Proof. The first assertion is obvious from the above construction. To prove the second, note that for $v \in L \cap \ker\Upsilon$ one has $\zeta(v) = 0$ because $-\iota_v\pi^*\gamma = 0$, where $\zeta(v)$ is defined by the formula $\zeta(v) = -\Upsilon^{-1}\left(\iota_{\pi(v)}\gamma\right)$. Since the subspace of $(0,1)$-vectors of the complex structure I_{Υ_t} varies by adding vectors from L, the projections of these vectors to the quotient V/L remain unchanged, so the $(0,1)$-subspace of V/L, and hence the induced complex structure, is independent on t, which proves the third assertion. □

Proposition 2.3. *Suppose that $X \xrightarrow{\pi} B$ is a Lagrangian fibration on a holomorphically symplectic manifold (X,Ω), and $\eta \in \Omega_{\mathrm{cl}}(B)$ a closed $(2,0)+(1,1)$-form on the base. Then the forms $\Omega_t = \Omega + t\pi^*\eta$ on X are complex symplectic, and this deformation (called* degenerate twistorial deformation*) preserves the Lagrangian fibration and the base.*

Proof. One can conclude from Proposition 2.1 that these forms are almost complex symplectic. As soon as the form η is closed, the forms Ω_t are closed, too, so they are complex symplectic by Proposition 1.7. The fibers stay Lagrangian, and the complex structure thereof, as well as of the base, remains unchanged by Proposition 2.2. □

For a hyperkähler manifold X the degenerate twistorial deformation produces an entire curve in the period space of X. In terms of the oriented 2-plane Grassmannian (see e. g. Section 3 in [**Dé**]), the plane corresponding to $(X, \Omega_{x+\sqrt{-1}y})$ is spanned by $(\Omega + \overline{\Omega}) + 2x\eta$ and $\sqrt{-1}(\Omega - \overline{\Omega}) - 2y\eta$. Thus one can define degenerate twistorial curves in an abstract situation: namely, for a 2-plane $\tau \in \mathrm{Gr}_{++}(V,q)$ and a vector $e \in V$ with $q(e,e) = 0$, the subvariety $\mathrm{Deg}_\tau(e) = \mathrm{Gr}_{++}(\mathrm{span}(\tau,e), q|_{\mathrm{span}(\tau,e)}) \subseteq \mathrm{Gr}_{++}(V,q)$ is an entire curve, and in the case when $V = H^2(X,\mathbb{R})$, q is the Bogomolov–Beauville–Fujiki form and $e = \pi^*[\omega]$ is the inverse image of the Kähler class on the base, this curve is exactly the base of the degenerate twistorial deformation.

Proposition 2.4. *Suppose $X \xrightarrow{\pi} B$ is a Lagrangian fibration on a holomorphically symplectic manifold (X,Ω), and $\sigma\colon B \to X$ is a smooth section (i. e. $\pi\circ\sigma = \mathrm{Id}_B$). Then there exists a deformation (X,Ω') of the holomorphic symplectic form s. t. the fibers stay Lagrangian, the complex structure on the fibers and the base stays the same, and in which this section σ is holomorphic.*

Proof. Consider the form $\eta = \sigma^*\Omega \in \Omega^2(B)$. By Proposition 1.6, it is of type $(2,0)+(1,1)$. Then by Proposition 2.3 the forms η gives rise to a deformation with desired properties. For $t = -1$ one has $\Omega_t|_{\sigma(B)} = (\Omega - \pi^*\sigma^*\Omega)\,|_{\sigma(B)} = \Omega|_{\sigma(B)} - \Omega|_{\sigma(B)} = 0$. By Proposition 1.3, this means that the submanifold $\sigma(B)$ is a complex submanifold, and since the projection is automatically a holomorphic map, the section σ is also holomorphic. □

3 Case of a K3 surface

In this section we assume that X is a K3 surface.

In terms of the Néron–Severi lattice elliptic fibrations on X correspond to primitive numerically effective classes $e \in \mathrm{NS}(X)$ with $(e,e) = 0$. Indeed, its linear system has no basepoints and establishes a map to $\mathbb{C}P^1$ with generic fiber elliptic curve. Cross-sections of the fibration determined by e correspond to classes $s \in \mathrm{NS}(X)$ with $(s,e) = 1$. Our first goal is to find an effective class $s \in H^2(X,\mathbb{Z})$ with $(s,e) = 1$ and a deformation in which s would have type $(1,1)$.

Proposition 3.1. *For any isotropic primitive vector e in an even unimodular lattice Λ there exists a vector $a \in \Lambda$ such that one has $(a,e) = 1$ and $(a,a) = -2$.*

Proof. Since the lattice Λ is unimodular, one can pick some vector $b \in \Lambda$ with $(b,e) = 1$. The number (b,b) is even since Λ is, so the vector $a = b-(1-(b,b)/2)\,e$ is integral. One has $(a,e) = (b,e)-(1+(b,b)/2)(e,e) = 1$ and $(a,a) = (b-(1+(b,b)/2)e, b-(1+(b,b)/2)e) = (b,b)-2(1+(b,b)/2)(b,e) = (b,b)-2-(b,b) = -2$. □

Proposition 3.2. *Suppose that X is a K3 surface with an elliptic fibration determined by an isotropic class e. Then the pullback of the Fubini–Study form from the base represents e.*

Proof. The Fubini–Study form on the base represents a Poincaré dual to the class of a point. Then its inverse image represents the Poincaré dual to the class of the inverse image of the point, i. e. of the fiber. □

Proposition 3.3. *Suppose that X is a K3 surface with elliptic fibration $X \xrightarrow{\pi} \mathbb{C}P^1$ determined by an isotropic class e, ω_{FS} stands for the Fubini–Study form on its base, and $s \in H^2(X,\mathbb{Z})$ is such that one has $(s,e) \neq 0$. Then we can deform the complex structure on X in such way that s belongs to $H^{1,1}(X)$.*

Proof. The forms $\Omega - t\pi^*\omega_{\text{FS}}$, $t \in \mathbb{C}$, are complex symplectic by Proposition 2.3. For such a form the condition on s to have type $(1,1)$ reads $(s, [\Omega - t\pi^*\omega_{\text{FS}}]) = 0$. Using the previous Proposition, one can rewrite this equation as $(s, [\Omega]) = t(s,e)$. It has solution in t whenever $(s,e) \neq 0$. ☐

Proposition 3.4. *Any elliptic fibration on a K3 surface possesses a deformation in which it admits a holomorphic section.*

Proof. Since the lattice $H^2(X,\mathbb{Z})$ for a K3 surface X is even and unimodular, by Proposition 3.1, one can find a vector $s \in H^2(X,\mathbb{Z})$ such that $(s,e) = 1$ and $(s,s) = -2$. By Proposition 3.3, we can deform the complex structure on X in such way that s would have type $(1,1)$. By Lefschetz theorem on $(1,1)$-classes, there exists a line bundle $L \to X$ with $c_1(L) = s$. Riemann–Roch theorem for this bundle L reads $\chi(L) = \chi(\mathscr{O}_X) + \frac{L\cdot(L\otimes K_X^*)}{2}$. By Serre's duality one has $h^2(L) = h^0(L^* \otimes K_X)$, and, as soon as K_X is trivial and $\chi(\mathscr{O}_X) = 2$, we can rewrite it as $h^0(L) - h^1(L) + h^0(L^*) = 2 + \frac{-2}{2} = 1$. Since L is nontrivial, either $h^0(L)$ or $h^0(L^*)$ does not vanish, and either L or L^* is effective. If L^* is, then $-s$ is represented by a curve, and $0 < (-s,e) = -(s,e) = -1$, hence L is effective and s is represented by a curve. Suppose that $s = \sum_i s_i$, where s_i are the classes of irreducible curves. One has $1 = (s,e) = (\sum_i s_i, e) = \sum_i (s_i, e)$. All the numbers (s_i, e) are positive integers with sum 1, so exactly one of them, say (s_0, e), equals 1. The curve represented by the class s_0 intersects each fiber at one point, i. e. is a section of the fibration. ☐

In particular, we prove that elliptic fibrations on K3 surfaces cannot have multiple fibers.

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