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Vacuum Geometry of the $N = 2$ Wess-Zumino Model

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Abstract: We give a mathematically rigorous construction of the moduli space and vacuum geometry of a class of quantum field theories which are $N = 2$ supersymmetric Wess-Zumino models on a cylinder. These theories have been proven to exist in the sense of constructive quantum field theory, and they also satisfy the assumptions used by Vafa and Cecotti in their study of the geometry of ground states. Since its inception, the Vafa-Cecotti theory of topological-antitopological fusion, or tt^* geometry, has proven to be a powerful tool for calculations of exact quantum string amplitudes. However, tt^* geometry postulates the existence of certain vector bundles and holomorphic sections built from the ground states. Our purpose in the present article is to give a mathematical proof that this postulate is valid within the context of the two-dimensional $N = 2$ supersymmetric Wess-Zumino models. We also give a simpler proof in the case of dimensional reduction to holomorphic quantum mechanics.

1. Introduction

The purpose of this paper is to provide a mathematically rigorous version of the physical theory of tt^* geometry, valid within constructive $N = 2$ quantum field theories.

In the setting of topological string theory, Witten [1] has shown that a partial understanding of background independence may be obtained from the geometry of theory space. tt^* geometry [2] is the theory of bundles, metrics, connections, and curvature over theory space. For Calabi-Yau spaces, this subject was studied by Strominger [3] and by Greene *et al.* [4] in the context of *special geometry*, which refers to the target-space geometry of $N = 2$ supersymmetric vector multiplets, possibly coupled to supergravity. Moreover, the seminal work [5] shows the importance of tt^* geometry as a powerful tool for calculations of exact quantum string amplitudes.

The ground state metric, originally introduced as a generalization of special geometry which is valid off-criticality in RG space, is a Hermitian metric on a complex vector bundle. The base space of this bundle is formed from suitable collections of coupling constants for the theory, while the fiber over a point in moduli space is built from the ground states of the associated quantum field theory. For the supersymmetric theories we study, the fibers may also be described as BRST cohomology of the supercharge operator.

One goal of this paper is to provide detailed descriptions of the coupling constant spaces relevant to the $N = 2$ Wess-Zumino model on a cylinder. Verification of the vector bundle axioms in these models is a quantum field theory version of the problem of continuity in κ of the Schrödinger operator $-\Delta + \kappa V$, thus existence of the vacuum bundle as a vector bundle in the rigorous sense requires analytic control over operator estimates. We also discuss the mathematical prerequisites necessary to define the metric and connection of tt^* geometry. A further mathematical question is the existence of a special gauge in which the anti-holomorphic components of the connection vanish.

1.1. Constructive Quantum Field Theory. We work with a class of quantum field theories which are two-dimensional Euclidean $N = 2$ Wess-Zumino models, making some technical assumptions which ensure that cluster expansion methods are valid. These interactions are also frequently called "Landau-Ginzburg" as the simplest bosonic self-interaction occurs in Landau and Ginzburg's study of condensed matter (see [25] for background).

We begin by defining the theory and recalling some known results. This is a theory of one complex scalar field ϕ and one complex Dirac fermion ψ. The formal Hamiltonian is given by

$$H = H_0 + \int \left(|W'_\lambda(\phi)|^2 - |\phi|^2 + \overline{\psi} \begin{bmatrix} W''_\lambda(\phi) - 1 & 0 \\ 0 & W''_\lambda(\phi)^* - 1 \end{bmatrix} \psi \right) dx \qquad (1)$$

where H_0 is the free Hamiltonian for a boson and fermion with unit mass and $W_\lambda(x) \equiv \lambda^{-2}\widetilde{W}(\lambda x)$. The cluster expansion is known to converge under the following conditions:

(A) $\widetilde{W}'$ must have $n-1$ distinct zeros $\xi_1, \ldots, \xi_{n-1}$, where $n = \deg \widetilde{W}$, and
(B) $|\widetilde{W}''(\xi_i)| = 1$ for all $i = 1, \ldots, n-1$.

The bosonic potential $|W'_\lambda(\phi)|^2$ has minima at the zeros of $\widetilde{W}'$ and scaling $\lambda \to 0$ increases the depth and the separation of the potential wells. Thus for sufficiently small λ, semiclassical analysis is valid.

The technical restrictions on our class of superpotentials make the theory amenable to cluster expansion methods, which have led to proofs of the existence of the infinite volume limit [7], and a vanishing theorem [8,10] for fermionic zero modes in the finite volume theory.

No new phenomena are expected for sufficiently small perturbations of the mass=1 condition (B). Moreover, for the present study, this condition must be removed; with condition (B) in place, the space of admissible potentials is not an open subset of the natural Euclidean space into which it is embedded.

As observed by Janowsky *et al.* [8], the relevant cluster expansions all continue to hold unchanged for small polynomial perturbations

$$W_\lambda(z) \longrightarrow \lambda^{-2}\widetilde{W}(\lambda z) + \lambda^{-1}\beta w(\lambda z) \qquad (2)$$

where $w \in \mathbb{C}[z]$ is a polynomial of degree n and β is a small parameter. This breaks any artificial symmetry due to the mass restriction, and enlarges the space of admissible superpotentials. Our approach to eliminating the mass restriction is to analyze this symmetry breaking in detail, and to show that the symmetry breaking perturbation (2) results in the replacement of the closed condition (B) with a condition that each $\widetilde{W}''(\xi_i)$ must lie in an appropriately small open neighborhood of the unit circle. See Sec. 5.1 and in particular, Theorem 6.

Imbrie *et al.* [9] studied the cluster expansion for the Dirac operator $i\partial\!\!\!/ + m(x)$ with a space-dependent mass, which is a toy model for the infinite volume multiphase $N = 2$ Wess-Zumino$_2$ theory. The appendix to [9] gives a method for removal of the $|m(x)| = 1$ restriction for the space-dependent Dirac operator, which assumably generalizes to the full Wess-Zumino model, giving a second method for removal of (B).

Integrating out the fermions gives the formal partition function

$$Z = \int d\mu(\phi) e^{-\int \left(|W'_\lambda(\phi)|^2 - |\phi|^2\right) dx} \det[1 + S\gamma_0 \chi_\Lambda (Y(W''_\lambda(\phi)) - 1)], \qquad (3)$$

with

$$Y(z) \equiv \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix}$$

where $d\mu(\phi)$ is the normalized Gaussian measure with covariance $(-\Delta + 1)^{-1}$. S is a fermionic propagator defined as $S = \gamma_0 (i\partial\!\!\!/ + 1)^{-1}$ where $\partial\!\!\!/ = \gamma^E_\mu \partial_\mu$, and γ^E are Euclidean gamma matrices defined by

$$-i\gamma_0 = \gamma^E_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^E_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The formal expression (3) is not well-defined without normal ordering. The normal-ordered partition function is

$$Z = \int d\mu(\phi) e^{-\int_\Lambda \left(:|W'_\lambda(\phi)|^2: - :|\phi|^2:\right) dx} \det{}_3[1 + K(\phi)] e^{-R}$$

where $K(\phi) = S\gamma_0 \chi_\Lambda (Y(W''_\lambda(\phi)) - 1)$ and R is a counter-term given by

$$R \equiv \int_\Lambda dx \left[|W'_\lambda(\phi)|^2 - : |W'_\lambda(\phi)|^2 : - |\phi|^2 + : |\phi|^2 : \right] + \frac{1}{2} \operatorname{Tr}(K^2(\phi)) - \operatorname{Tr} K(\phi)$$

Supersymmetry of the theory implies that the counter-term R is *finite*, which means that if we regularize R then the limit as the regularization is removed is well-defined. In finite volume this theory was constructed in [22] and [23] with no restriction on the superpotential. The infinite volume limit is treated via cluster expansions in [7] and [24].

1.2. Supersymmetric Lagrangians. The transformation properties of the Wess-Zumino model under supersymmetry become especially transparent when it is written in terms of a manifestly supersymmetric action,

$$S = \int d^4\theta K(\Phi, \overline{\Phi}) + \int d^2\theta^+ W(\Phi) + \int d^2\theta^- \overline{W}(\overline{\Phi}) \qquad (4)$$

where Φ is a superfield.

Typically in constructive field theory one restricts attention to the Kähler form

$$(-\frac{1}{4}\Phi^*\Phi)$$

arising from a flat metric, since some work is required to generalize the cluster expansion to more general K. We hope to address this question in a separate paper, but for the present we also use the flat Kähler form.

Expanding the superfields in lightcone coordinates and eliminating auxiliary fields from the Lagrangian density (4) using their equations of motion, one obtains

$$\mathcal{L} = \sum_{i=1}^{n} \Big(\frac{1}{2}\partial_+\varphi_i^*\partial_-\varphi_i + \frac{1}{2}\partial_-\varphi_i^*\partial_+\varphi_i - |\partial_i W(\varphi)|^2 + i\psi_{1,i}^*\partial_-\psi_{1,i} + i\psi_{2,i}^*\partial_+\psi_{2,i} - (\sum_{j=1}^{n}\psi_{1,i}\psi_{2,j}^*\partial_i\partial_j W(\varphi) + \text{h.c.}) \Big) , \quad (5)$$

It is clear that the Hamiltonian (1) is of the type obtained by applying a Legendre transformation to the on-shell Lagrangian (5).

1.3. Topological Twisting. One may couple a Landau-Ginzburg theory to an arbitrary $U(1)$ gauge field A_μ, so that the correlator $\langle \prod \mathcal{O} \rangle$ of an arbitrary product of local operators $\mathcal{O}$ depends on A_μ as well as the spin connection ω_μ. This coupling introduces an extra term into the Lagrangian, given by

$$A_z(\psi_-)^i(\psi_-)_i + A_{\overline{z}}(\psi_+)^i(\psi_+)_i. \quad (6)$$

When A is set equal to $\frac{1}{2}$ times the spin connection, a field which previously had spin s and fermion charge q will now have spin $s - \frac{1}{2}q$. In particular, Q_+ which had spin $1/2$ and fermion number $+1$ becomes a scalar.

With operators $\mathcal{O}$ set equal to chiral primary fields and with $A_\mu \sim \frac{1}{2}\omega_\mu$, the correlator

$$\Big\langle \prod \mathcal{O} \Big\rangle_{A_\mu,\omega_\nu}$$

is a topological invariant. In particular one computes from (6) that gauging with the connection

$$A_z = -\frac{i}{2}\omega_z, \quad A_{\overline{z}} = +\frac{i}{2}\omega_{\overline{z}}$$

has the effect of modifying the stress energy tensor,

$$T_{ab} \;\longrightarrow\; T'_{ab} \equiv T_{ab} - \frac{1}{2}\varepsilon_a{}^c\partial_c J_b \quad (7)$$

The new stress energy tensor is BRST exact, in the sense that $\exists\ \Lambda_{ab}$ such that $T'_{ab} = \{Q, \Lambda_{ab}\}$, where $Q = Q_+ + \overline{Q}_-$. Observables in the topological theory are identified with BRST exact objects. Any theory in which the action is supersymmetric and the stress-energy tensor is a Q-commutator is topological, since by definition the stress-energy tensor generates metric deformations.

1.4. The tt^* Equations. Physical observables of an $N = 2$ SCFT are associated with chiral superfields, with components

$$\Phi_i = (\phi_i^{(0)}(z,\overline{z}), \phi_i^{(1)}(z,\overline{z}), \overline{\phi}_i^{(1)}(z,\overline{z}), \phi_i^{(2)}(z,\overline{z})) \tag{8}$$

where

$$\phi_i^{(2)} = \{Q^-, [\overline{Q}^-, \phi_i^{(0)}]\} \tag{9}$$

We define a deformed theory parameterized by the coupling constants $(t_i, \bar{t}_{\bar{i}})$ as follows

$$\mathcal{L}(t_i, \bar{t}_i) = \mathcal{L}_0^{N=2} + \sum_i t_i \int_\Sigma \phi_i^{(2)} + \sum_{\bar{i}} \bar{t}_{\bar{i}} \int_\Sigma \overline{\phi}_{\bar{i}}^{(2)} \tag{10}$$

This deformed theory can be transformed into a TFT by the twisting mechanism. If some of the non vanishing coupling constants correspond to relevant deformations, then the theory defined by (10) will represent a massive deformation of the $N = 2$ SCFT defined by $\mathcal{L}_0^{N=2}$.

Let $|i, t, \bar{t}; \beta\rangle$ be the state defined by inserting on the hemisphere the field ϕ_i and projecting on a zero energy state by gluing the hemisphere to an infinitely long cylinder of perimeter β. This corresponds to using a metric $g = e^\phi dz d\overline{z}$ with $\beta = e^\phi$. Let us now introduce a set of connection forms A_i, $A_{\bar{i}}$. These connections are defined by

$$\langle \overline{k} | \partial_i - A_i | j \rangle = 0 \tag{11}$$

with $|\overline{k}\rangle$ the antiholomorphic basis. An alternate definition is in terms of the hemisphere states,

$$\begin{aligned} \partial_{t_i} |j, t, \bar{t}; \beta\rangle &= A_{ij}^k |k, t, \bar{t}; \beta\rangle + Q^+\text{-exact} \\ \partial_{\bar{t}_i} |j, t, \bar{t}; \beta\rangle &= A_{\bar{i}j}^k \; |k, t, \bar{t}; \beta\rangle + Q^+\text{-exact} \end{aligned} \tag{12}$$

The connection (11) is related to (12) by

$$A_{ij}^k = A_{ij\overline{k}} g^{\overline{k}k}$$

with $g^{\overline{k}k}$ the inverse of the hermitian metric $g_{i\bar{j}} = \langle \bar{j} | i \rangle$.

Therefore the covariant derivatives are given by

$$D_i = \partial_i - A_i \, , \qquad \overline{D}_{\bar{i}} = \partial_{\bar{i}} - A_{\bar{i}} \tag{13}$$

Using the functional integral representation of $|i, t, \bar{t}\rangle$ and interpreting the partial derivative ∂_i as the insertion and integration over the hemisphere of the operator $\phi_i^{(2)}$, we can conclude by contour deformation techniques that $\partial_i |j, t, \bar{t}; \beta\rangle$ is also a physical state and $A_{\bar{i}j}^k = 0$.

Defining now

$$A_{ijk} = \langle k | \partial_i | j \rangle = A_{ij}^l \; \eta_{lk} \tag{14}$$

for η_{lk} the topological metric, standard functional integral arguments give curvature equations for the connections A_i. In particular,

$$\partial_{\bar{l}} A_{ij}^k = \beta^2 [\, C_i, \overline{C}_{\bar{l}} \,]_j^k \tag{15}$$

with β the perimeter of the cylinder, and C_i are chiral ring structure constants. Equation (15), first derived by Cecotti and Vafa [2], together with

$$[D_i, D_j] = [\overline{D}_{\bar{i}}, \overline{D}_{\bar{j}}] = [D_i, \overline{C}_{\bar{j}}] = [\overline{D}_{\bar{i}}, C_j] = 0$$

$$[D_i, C_i] = [D_j, C_i] \ \ , \ \ [\overline{D}_{\bar{i}}, \overline{C}_{\bar{j}}] = [\overline{D}_{\bar{j}}, \overline{C}_{\bar{i}}]$$

which can be deduced by similar techniques, are known as the tt^* equations. To contract the indices of the topological and antitopological structure constants, one must use the metric $g_{i\bar{j}}$ of the physical Hilbert space $\mathcal{H}$.

2. Holomorphic Analysis in Banach Spaces

We now begin the mathematically rigorous portion of the paper. We briefly review classical theory of analytic mappings between (possibly infinite dimensional) Banach spaces and some aspects of Schrödinger operators, and prove results which are a generalization of Kato-Rellich theory. We then present a treatment of vacuum bundle theory for Schrödinger operators, which is a new way of analyzing this class of operators.

2.1. Holomorphic Families of Unbounded Operators. A family of bounded operators $T(\chi) \in \mathcal{B}(X, Y)$ between two Banach spaces is said to be *holomorphic* if it is differentiable in norm for all χ in a complex domain. For applications, it is not sufficient to consider bounded operators only, and the notion of holomorphy needs to be extended to unbounded operators. $D(\,\cdot\,)$ denotes the domain of an operator, and $\rho(\,\cdot\,)$ denotes the resolvent set.

Definition 1. *A family of closed operators $T(\chi) : X \to Y$ defined in a neighborhood of $\chi = 0$, where X, Y are Banach spaces, is said to be* holomorphic at $\chi = 0$ *if there is a third Banach space Z and two families of bounded operators $U(\chi) : Z \to X$ and $V(\chi) : Z \to Y$ which are bounded-holomorphic at $\chi = 0$ such that $U(\chi)$ is a bijection of Z onto $D(T(\chi))$, and $T(\chi)U(\chi) = V(\chi)$.*

An equivalent condition which is easier to check in some cases involves holomorphicity for the resolvent in the usual sense of bounded operators. We have [18]:

Theorem 1. *Let $T(\chi)$ be a family of closed operators on a Banach space X defined in a neighborhood of $\chi = 0$, and let $\zeta \in \rho(T(0))$. Then $T(\chi)$ is holomorphic at $\chi = 0$ if and only if for all χ in some small ball, $\zeta \in \rho(T(\chi))$ and $R(\zeta, \chi) := (T(\chi) - \zeta)^{-1}$ is bounded-holomorphic. In this situation, $R(\zeta, \chi)$ is jointly bounded-holomorphic in two variables.*

An interesting variant of this (which will arise in the case of interest for this paper) is the following.

Definition 2. *A family $T(\chi)$ of closed operators from X to Y defined for χ in a domain Ω in the complex plane is said to be* holomorphic of type (A) *if*

1. *The domain $D := D(T(\chi))$ is independent of $\chi \in \Omega$.*
2. *For every $u \in D$, $T(\chi)u$ is holomorphic for $\chi \in \Omega$.*

A family that is type (A) is automatically holomorphic in the sense of Definition 1, taking Z to be the Banach space D with the norm $\|u\|_Z := \|u\| + \|T(0)u\|$. We now consider analytic perturbations of the spectrum. The following Theorem from Ref. [18] will be used in the proofs of our main results.

Theorem 2. *Let X be a Banach space and $T(\chi) \in \mathcal{C}(X)$ be holomorphic in χ near $\chi = 0$ and let $\Sigma(0) = \Sigma(T(0))$ be separated into two parts $\Sigma'(0), \Sigma''(0)$ in such a way that there exists a rectifiable simple closed curve Γ enclosing an open set containing Σ' in its interior and Σ'' in its exterior. In this situation, for $|\chi|$ sufficiently small, $\Sigma(T(\chi))$ is also separated by Γ into two parts $\Sigma'(T(\chi)) \cup \Sigma''(T(\chi))$, and X decomposes as a direct sum $X = M'(\chi) \oplus M''(\chi)$ of spectral subspaces. Moreover, the projection on $M'(\chi)$ along $M''(\chi)$ is given by $P(\chi) = -\frac{1}{2\pi i}\oint_\Gamma R(\zeta,\chi)d\zeta$ and is bounded-holomorphic near $\chi = 0$.*

Remark on Lemma 2. The projection $P(\chi)$ is called the *Riesz projection*, and this projection being bounded-holomorphic is equivalent to the statement that the subspaces $M'(\chi)$ and $M''(\chi)$ are holomorphic in their dependence on χ.

2.2. Perturbation Theory. Consider a family of closed operators T_χ depending on a parameter $\chi \in B_\varepsilon(0)$ for some $\varepsilon > 0$, with a common domain D in a Hilbert space $\mathcal{H}$, and such that each T_χ has a nonempty resolvent set. Write $T_\chi = T_0 + V_{\text{eff}}(\chi)$, where $V_{\text{eff}}(\chi) := T_\chi - T_0$ is called the effective potential.

Definition 3. *A discrete eigenvalue λ of T_0 is said to be* stable *with respect to V_{eff} if*

1. *$\exists\, r > 0$ s.t. $\Gamma_r \equiv \{|z-\lambda| = r\} \subset \rho(T_\chi)$ for all $|\chi|$ sufficiently small, and*
2. *$P(\chi) \equiv -\frac{1}{2\pi i}\oint_{\Gamma_r}(T_\chi - \zeta)^{-1}d\zeta$ converges to $P(0)$ in norm as $\chi \to 0$.*

The notion of stability arises in the following rigorous statement of degenerate perturbation theory, which is adapted from results of Kato. Here, $m(\lambda)$ denotes multiplicity of eigenvalue λ.

Theorem 3 (Degenerate Perturbation Theory). *Let T_χ be a Type (A) family near $\chi_0 = 0$. Let λ_0 be a stable eigenvalue of T_0. Then there exist families $\lambda_\ell(\chi),\ \ell = 1\dots r$, of discrete eigenvalues of T_χ such that*

1. *$\lambda_\ell(0) = \lambda_0$ and $\sum_{\ell=1}^r m(\lambda_\ell(\chi)) = m(\lambda_0)$.*
2. *Each $\lambda_\ell(\chi)$ is analytic in $\chi^{1/p}$ for some $p \in \mathbb{Z}$, and if T_χ is self-adjoint $\forall \chi \in \mathbb{R}$, then $\lambda_\ell(\chi)$ is analytic in χ.*

3. The Vacuum Bundle for Schrödinger Operators

The free Schrödinger operator $P^2 = -\Delta$ in d space dimensions is self-adjoint on the domain $D(P^2) = H^2(\mathbb{R}^d)$, and has $C_0^\infty(\mathbb{R}^d)$ as a core. Our goal in this section is to consider perturbations V_χ of H_0 which depend analytically on (coupling) parameters χ, and to show that under certain reasonable classes of such perturbations, the total Schrödinger operator $P^2 + V_\chi$ remains self-adjoint and has the appropriate spectral splitting condition to apply holomorphic Kato theory. Ultimately this leads to the rigorous construction of a vacuum bundle for quantum mechanics, which is used later for vacuum estimates in the more complicated Wess-Zumino field theory model.

There are a number of conditions on a potential V which guarantee that the Schrödinger operator $P^2 + V$ will be essentially self-adjoint. An example on $\mathbb{R}^3$ of one such condition is the following. Let R denote the family of potentials $f(x)$ on $\mathbb{R}^3$ obeying

$$\int \frac{|f(x)|\ |f(y)|}{|x-y|^2}\, dxdy < \infty.$$

Then $V \in L^\infty(\mathbb{R}^3) + R \ \Rightarrow\ P^2 + V$ is essentially selfadjoint [17].

Definition 4. *We will refer to a function space W as a* space of admissible potentials *if $\forall\ f \in W$, the Schrödinger operator $P^2 + f$ is essentially self-adjoint.*

Standard self-adjointness theorems for Schrödinger operators generically tend to have the property that the space W of all admissible potentials is a locally convex space. A locally convex topological vector space is the minimal structure which is necessary for the traditional definition of "holomorphic map" to remain valid with no modifications. A map $T : U \to W$ from a domain $U \subset \mathbb{C}$ into a locally convex space W is said to be *holomorphic* at $z_0 \in U$ if $\lim_{z\to z_0} \frac{T(z)-T(z_0)}{z-z_0}$ exists. These definitions allow us to speak of a holomorphic map V from a complex manifold M into a space W of admissible potentials. This generalizes to a geometric setting the notion of a perturbation which depends on a number of coupling parameters; in our case coordinates on M take the role of generalized couplings.

Theorem 4. *Let $U \subset X$ be an open connected set in a Banach space X and let $\mathcal{H}$ be a Hilbert space. Let H_0 be a closed operator on a dense domain $\mathcal{D} \subseteq \mathcal{H}$. Fix a map $V : U \to Op(\mathcal{H})$, and for $\tau \in U$, define $H(\tau) = H_0 + V(\tau)$, which we assume has nonempty resolvent set. Assume $\forall\tau$, $V(\tau)$ has H_0-bound smaller than one, and that $V(\tau)\psi$ is analytic in τ, for any $\psi \in \mathcal{D}$. Then $H(\,\cdot\,)$ is analytic.*

Proof of Theorem 4. By the Kato stability theorem [18], $H(\tau)$ is closed for all τ. Since $V(\tau)$ is H_0-bounded, $D(H(\tau)) = D(H_0) \cap D(V(\tau)) = D(H_0)$. It follows that the family $H(\,\cdot\,)$ is type (A), and hence analytic. □

Our assumptions in Theorem 4 are sufficiently general to allow the domain of the map V to be an arbitrary manifold.

If we assume H_0 to be a selfadjoint operator on a dense domain $\mathcal{D} \subset L^2(\mathbb{R}^n)$, and we let V_i for $i \in \mathbb{N}$ be a sequence of uniformly bounded operators on $L^2(\mathbb{R}^n)$ and $\tau \in \ell^\infty(\mathbb{C})$, then Theorem 4 implies that the Hamiltonian

$$H(\tau) = H_0 + \sum_{i=1}^{\infty} \tau_i V_i$$

is analytic in the coupling parameters τ_i.

In order to apply the Remark following Lemma 2, we need to work in a scenario where the lowest eigenvalue of the Schrödinger operator is an isolated eigenvalue. This is by no means guaranteed; in fact it is typically false on $L^2(\mathbb{R}^n)$ when $V(x)$ is continuous and $\lim_{x\to\infty} V(x) = 0$. However, this spectral gap is guaranteed given a compact manifold and some very generic conditions on V, and on a noncompact manifold such as $\mathbb{R}^d$ when $V(x)$ grows at infinity. We discuss both the compact and non-compact cases since the non-compact case is usually studied in quantum mechanics, but quantum field theory is frequently studied on a compact manifold.

Lemma 1. *Define $H = -\Delta + V(x)$ on $L^2(X)$ for a compact Riemannian manifold X, and assume that $V \in L^2(X)$ with $V(x) \geq 0$. Then H has purely discrete spectrum in which the eigenvalues are not bounded above, and all eigenvalues have finite algebraic multiplicity.*

The proof of Lemma 1 uses standard methods along the lines of Griffiths and Harris' proof of the Hodge theorem [15]. Lemma 1 implies a spectral gap between the lowest eigenvalue (ground state) of H and the first excited state eigenvalue on a compact manifold.

Generally, if the resolvent $R_H(z)$ is compact, then $\sigma(R_H(z))$ is discrete with 0 the only possible point in σ_{ess}. Hence one would expect that H has discrete spectrum with only possible accumulation point at ∞, and this implies $\sigma_{ess}(H) = \emptyset$. This reasoning shows that if $V(x) \geq 0$, V is in $C(\mathbb{R}^n)$ or $L^2_{loc}(\mathbb{R}^n)$, and $V(x) \to \infty$ as $\|x\| \to \infty$, then $H = -\Delta + V$ has purely discrete spectrum on $L^2(\mathbb{R}^d)$.

If $z_0 \in \sigma(T_0)$ is an N-fold degenerate eigenvalue of T_0, then generically a perturbation will break the degeneracy, and therefore, if T_κ is a holomorphic perturbation of T_0 we expect, as in Theorem 3, a number of eigenvalue curves which flow away from z_0. It follows that we have a vacuum bundle only in the special cases when the degeneracy N is not broken by T_κ, for all κ lying in some complex manifold. Physics intuition suggests the only way this can happen is in the presence of additional symmetry, such as supersymmetry. In the latter case, the Witten index [12], which counts the ground states weighted by $(-1)^F$, equals the index of the Dirac operator and this does not change for all effective superpotentials of the same degree.

The following is the main theorem of Section 3. It asserts the existence of the vacuum bundle for a Schrödinger operator.

Theorem 5. *Let M be either a finite dimensional complex manifold or an infinite-dimensional complex Banach manifold, and let X be a finite dimensional real manifold with a Riemannian metric. Let Y be a linear space whose elements are complex-valued functions on X, such that for any $f \in Y$,*

1. *The multiplication operator f on $L^2(X)$ is P^2-bounded with P^2-bound < 1.*
2. *$P^2 + f$ has spectral gap between first and second eigenvalues on $L^2(X)$.*

Let $V : M \to Y$ be holomorphic, and for $\tau \in M$ let $H_\tau := P^2 + V_\tau$ have lowest energy $\lambda_0(\tau)$ with eigenspace $E_0(\tau)$. If $\dim E_0(\tau)$ is constant, then $E_0 \to M$ is a holomorphic vector bundle.

Proof of Theorem 5. Since $V(\tau)$ is P^2-bounded with P^2-bound < 1, the Kato-Rellich theorem implies that for any $\tau \in M$, $H(\tau) = P^2 + V(\tau)$ is self-adjoint on $\mathcal{D} = D(P^2)$. For $\psi \in \mathcal{D}$, $V(\tau)\psi$ has the form of $f(\tau, x)\psi(x)$ as a function on $x \in X$, where $f(\tau, x)$ is analytic in τ by assumption. We conclude by Theorem 4 that $H(\tau)$ is analytic. To show that the ground state subspace is analytic, we work with operators having discrete spectrum with spectral gap (see Lemma 1 and the discussion thereafter). We may therefore apply the Remark following Lemma 2. Since $\dim E_0(\tau)$ is constant, it follows that we may choose N holomorphic functions $v_i(\tau), i = 1 \ldots N$ s.t. $\forall\, \tau$, $\{v_i(\tau)\}$ form a linearly independent spanning set of $E_0(\tau)$. $\square$

Remark 1. The space Y has to be tuned to the space X so that conditions 1 and 2 in the theorem are satisfied. For example, if $X = \mathbb{R}^d$, then Y can consist of elements of $C(\mathbb{R}^d)$ or $L^2_{loc}(\mathbb{R}^d)$ that blow up at infinity. If X is a compact manifold, then we can take $Y = \{f \in L^2(X) : f(x) \geq 0 \; \forall x \in X\}$. This suggests a general class of new problems in functional analysis. Given X, the problem is to determine the largest space Y which is tuned to X in the sense of Theorem 5.

4. The Wess-Zumino Model, the Dirac Operator on Loop Space, and Vanishing Theorems

4.1. The Wess-Zumino Model on a Cylinder. In a fundamental paper [21], Jaffe, Lesniewski, and Weitsman present rigorous results for supersymmetric Wess-Zumino models by generalizing index theory of Dirac operators to an infinite dimensional setting; we now give a concise introduction to the results of [21] and recall a number of facts from constructive field theory which will be needed in later sections.

We study self-adjoint Hamiltonians H defined on the Hilbert space $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$, where $\mathcal{H}_b$ and $\mathcal{H}_f$ are, respectively, the symmetric and antisymmetric tensor algebras over the one-particle space $W = W_+ \oplus W_-$, where W_+ and W_- represent single particle/antiparticle states respectively, and $W_\pm \equiv L^2(T^1)$. The Hamiltonian is that corresponding to one massive complex (Dirac) fermion field ψ of mass m, and one complex boson field φ with the same mass as the fermion field, defined on a circle of length ℓ. The interactions are parameterized by a holomorphic polynomial $V(z)$, known as the *superpotential*. The free Hamiltonian in second-quantized notation is written as

$$H_0 = \sum_{j=\pm,\ p\in\hat{T}^1} \omega(p) \left(a_j^*(p)a_j(p) + b_j^*(p)b_j(p)\right) ,$$

where a_j satisfy canonical commutation relations for bosonic oscillators, and b_j satisfy the corresponding Fermion algebra.

We can write the superpotential as $V(\varphi) = \frac{1}{2}m\varphi^2 + P(\varphi)$, separating out the mass term. The energy density of the bosonic self interaction is $|\partial V(\varphi)|^2$, a polynomial of degree $2n-2$. The boson-fermion interaction is known as a generalized Yukawa interaction, and has the form

$$\overline{\psi}\Lambda_+\psi\partial^2 V + \overline{\psi}\Lambda_-\psi(\partial^2 V)^*,$$

where $\Lambda_\pm$ are projections onto chiral subspaces of spinors. If $P = 0$, this interaction reduces to a free mass term $m\overline{\psi}\psi$.

Define operators $N_{\tau,\{b,f\}}$ by

$$N_{\tau,b} = \sum_{j=\pm,p\in\hat{T}^1} \omega(p)^\tau a_j^*(p)a_j(p), \qquad N_{\tau,f} = \sum_{j=\pm,p\in\hat{T}^1} \omega(p)^\tau b_j^*(p)b_j(p).$$

Then the family of operators $N_\tau = N_{\tau,b} \otimes I + I \otimes N_{\tau,f}$ interpolates between the total particle number operator N_0 and the free Hamiltonian N_1. We write N_f for $N_{0,f}$. A selfadjoint unitary operator that is not the identity necessarily has $+1$ and -1 eigenvalues, and is therefore a $\mathbb{Z}_2$-grading. $\Gamma = \exp(i\pi N_f)$ is self-adjoint and unitary, hence the Hilbert space splits into a direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of the ± 1 eigenspaces of Γ, and thus naturally inherits the structure of a super vector space.

The following bilinear form over $\mathcal{H}$ is known as the *supercharge*:

$$Q = \frac{1}{\sqrt{2}} \int_{T^1} dx\; \psi_1(\pi - \partial_1\varphi^* - i\partial V(\varphi)) + \psi_2(\pi^* - \partial_1\varphi - i\partial V(\varphi)^*) + h.c. \quad (16)$$

where the superpotential $V(\varphi) = \frac{1}{2}m\varphi^2 + \sum_{j=3}^{n} a_j\varphi^j$ is a holomorphic polynomial with $n \geq 3,\ a_n \neq 0$, and $m > 0$. With appropriate regularization and limiting procedures, we have $H = Q^2$, where H is the full interacting Hamiltonian.

Define $\mathcal{D}(T^1)$ as the space of smooth maps $T^1 \rightarrow \mathbb{C}$, with topology defined by uniform convergence of each derivative. $\mathcal{D}(T^1)$ is an infinite-dimensional Fréchet manifold known as *loop space,* and Q has the structure of a Dirac operator on loop space. The proof that the bilinear form (16) defines an operator requires careful analysis, which has been done in [21]. The strategy is to split the expression (16) for Q into a free part and an interacting part, and to further regularize the interacting part by convolving the fields $\varphi(x), \psi_\mu(x)$ with a smooth approximation to the periodic Dirac measure, which implements a momentum space cutoff.

To obtain the desired approximation to periodic Dirac measure, we use a cutoff function χ satisfying: $0 \leq \chi \in \mathcal{S}(\mathbb{R})$, $\int_{-\infty}^{\infty} \chi(x)dx = 1$, $\chi(-x) = \chi(x)$, $\hat{\chi}(p) \geq 0$, $\operatorname{supp} \hat{\chi}(p) \subset [-1, 1]$, and $\hat{\chi}(p) > 0$ for $|p| \leq 1/2$. We set

$$\chi_\kappa(x) = \kappa \sum_{n\in\mathbb{Z}} \chi(\kappa(x - n\ell))$$

where $\kappa > 0$. Regularized (cutoff) fields are defined by taking convolution with χ_κ on T^1,

$$\varphi_\kappa(x) = \chi_\kappa * \varphi(x),\ \ \psi_{\mu,\kappa}(x) = \chi_\kappa * \psi_\mu(x)\,.$$

The result of this procedure is a regularized supercharge $Q(\kappa) = Q_0 + Q_{i,\kappa}$. *A priori* estimates [22] establish a homotopy between $Q(\infty)$ and $Q(0)$ with $i(Q_+(\kappa))$ constant. Explicit calculation[21] shows that $Q_0^0 + Q_{i,0}$ is the supercharge of the model of $N = 2$ holomorphic quantum mechanics considered in [20] and this paper. Existence of a holomorphic vacuum bundle for the quantum mechanical supercharge $Q_0^0 + Q_{i,0}$ follows by dimensional reduction from Theorem 7. However the holomorphic quantum mechanics model is sufficiently simple that the desired vacuum bundle estimates can be established directly using methods of classical ODEs, as we show in Section 7.

It was shown in [20] that $Q(0)$ has only bosonic ground states, i.e. $n_-(Q(0)) = 0$. We say that a Hamiltonian has the *vanishing property* if $n_- = 0$.

4.2. $N = 2$ Wess-Zumino$_2$ Vanishing Theorem. We recall the vanishing theorem for $N = 2$ Wess-Zumino$_2$ models, independently proven by Janowsky and Weitsman [8], and by Borgs and Imbrie [10], which is crucial for later sections. Consider superpotentials of the form

$$\lambda^{-2}\widetilde{W}(\lambda x) + \lambda^{-1}\xi w(\lambda x) \tag{17}$$

where $\widetilde{W}$ and w are polynomials of degree n, $\widetilde{W}'$ has $n-1$ distinct zeros, and $|\widetilde{W}''| = 1$ at each zero. The $N = 2$ Wess-Zumino quantum field theories corresponding to superpotentials of type (17) have no fermionic zero modes for λ and ξ sufficiently small, where λ is a parameter that controls the depth and spacing of the potential wells, and ξ measures the strength of w, which represents a small perturbation away from the unit mass condition.

To see this, we note that results of [22,23] imply that $e^{-\tau H}$ is trace class for all $\tau > 0$ and $\operatorname{ind}(Q) = \operatorname{tr}(\Gamma e^{-\tau H}) = \deg(V) - 1$. It follows that

$$\dim\ker H = \lim_{\beta\to\infty} \operatorname{Tr}(e^{-\beta H}),$$

and given the assumptions on λ and ξ, cluster expansion methods [8] show that for β sufficiently large,

$$\left|\mathrm{Tr}(\Gamma e^{-\beta H}) - \mathrm{Tr}(e^{-\beta H})\right| < \frac{1}{2} \tag{18}$$

Now Q is selfadjoint, $H = Q^2 \geq 0$ and [23] shows that $e^{-\beta H}$ is trace class, hence

$$\dim\ker H = \mathrm{Tr}(e^{-\beta H}) + O(e^{-\beta\varepsilon})$$

for $\beta \gg 1$ and for some $\varepsilon > 0$. It now follows from (18) that

$$|\dim\ker H - \mathrm{ind}(Q)| < 1.$$

In this situation, $\dim\ker H$ and $\mathrm{ind}(Q)$ are integers differing by less than one, hence they are equal. It follows that for superpotentials as in (17), $n_-(H) = 0$.

4.3. Other vanishing theorems. Some care is required, as the term 'vanishing theorem' can take on other, perhaps contradictory, interpretations. For example, if M is a compact spin manifold with a nontrivial S^1-action, Atiyah and Hirzebruch [14] have shown that $\mathrm{Ind}(D) = \hat{A}(M) = 0$, where D is the Dirac operator on M. In a situation more closely related to quantum field theory, Witten [13] formally applied the Atiyah-Bott-Segal-Singer fixed point formula to the Dirac operator D^L on loop space LM, with the result that, with M as above and under suitable assumptions on the first Pontryagin class, the Witten genus $\mathrm{Ind}(D^L) = 0$. In the present context, $\mathrm{Ind}\, Q = 0$ entails $n_-(Q) = n_+(Q)$ and does not imply that the zero modes are purely bosonic, so the Janowsky-Weitsman theorem is a qualitatively different result from Witten's vanishing theorem, applied with Q playing the role of a Dirac operator.

We will show that the vacuum bundle exists for $N = 2$ models with the vanishing property. A large class of Wess-Zumino models (precisely those with superpotentials of the form (17)), are known to have the $n_- = 0$ property. We conjecture that a vanishing theorem stronger than [8] holds, and that all $N = 2$ Wess-Zumino models on a cylinder satisfy $n_- = 0$.

It is interesting to note that the vanishing theorem of Janowsky-Weitsman [8] and Borgs-Imbrie [10] is expected *not* to hold for the corresponding $N = 1$ Wess-Zumino models. In [20], a quantum mechanics version of the $N = 1$ Wess-Zumino field theory is considered. Supersymmetry is broken or unbroken depending on the asymptotics of the superpotential at infinity, and is characterized by its degree: $i(Q_+) = \pm\ \deg\ V (\mathrm{mod}\ 2)$. In the unbroken case, there is a unique ground state; it belongs to $\mathcal{H}_+$ $(n_+ = 1,\ n_- = 0)$ or to $\mathcal{H}_-$ $(n_+ = 0,\ n_- = 1)$, according to the additional $\mathbb{Z}_2$ symmetry of the superpotential. In the case of broken supersymmetry, there are exactly two ground states and $n_+ = n_- = 1$. Similar results are true in the corresponding $d = 2$ quantum field models in a finite volume [21].

Thus the vanishing property is an aspect of $N = 2$ supersymmetry, as is the theory of the ground state metric, tt^* geometry, and the CFIV index [6,2].

4.4. The Vacuum Bundle and Atiyah-Singer Index Theory. Let $\mathcal{C}(\mathcal{H})$ denote the space of closed unbounded operators on Fock space $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$. Suppose that we have identified the appropriate moduli space $\mathcal{M}$ of coupling constants for a supersymmetric quantum field theory with supercharge Q and Hamiltonian H. For example, the space

$\mathcal{JW}$ introduced our construction of the vacuum bundle is such a space (although not the largest) for $N = 2,\ n_- = 0$ Wess-Zumino theories.

In view of the theory developed in Sections 4 and 5.1, quantum field theory provides a map from the total moduli space $\mathcal{M}$ into $\mathcal{C}(\mathcal{H})$, given by associating the supercharge operator $Q_\mathcal{T}$ to any set of coupling constants $\mathcal{T} \in \mathcal{M}$. Composing this map with the squaring function gives the Hamiltonian of the theory also as a map $\mathcal{M} \longrightarrow \mathcal{C}(\mathcal{H})$, defined by $\mathcal{T} \to (Q_\mathcal{T})^2 \equiv H_\mathcal{T}$. This induces a map from $\mathcal{M} \to Gr(\mathcal{H})$ given by associating $\mathcal{T} \to \ker H_\mathcal{T}$, where $Gr(\mathcal{H})$ denotes the Grassmannian of closed subspaces of $\mathcal{H}$, with topology given by identifying closed subspaces with projectors and imposing a standard operator topology.

The vanishing property is the statement that

$$\dim \ker\ H_\mathcal{T}|_{\mathcal{H}_-} = 0 \ \text{ for all } \ \mathcal{T} \in \mathcal{M}$$

where $\mathcal{H}_-$ denotes the -1 eigenspace (or *fermionic subspace*) of the $\mathbb{Z}_2$-grading operator Γ.

Let $D : \Gamma(E) \longrightarrow \Gamma(F)$ be an elliptic operator and let E and F be vector bundles over a closed manifold M. The Atiyah-Singer Index Theorem states that

$$\operatorname{Ind} D := \dim \operatorname{Ker} D - \dim \operatorname{Coker} D = \langle P(M, \sigma_{\rm top}(D)), [M]\rangle\,.$$

The quantity on the right is a characteristic number built from the topology of M and topological information contained in the top order symbol of D.

Atiyah and Singer also proved the Families Index Theorem, which applies to a family of elliptic operators D_n for n ranging in a compact manifold N. The Families Index Theorem identifies the Chern character of the index bundle $\underline{\rm Ind}(D)$ in $H^*(N;\mathbb{Q})$ with a characteristic class on N built from the topology of N and the pushforward of the symbols of the operators D_n. The index bundle is a virtual bundle whose fiber for generic $n \in N$ is the formal difference $\operatorname{Ker}(D_n) - \operatorname{Coker}(D_n)$, i.e.

$$\underline{\rm Ind} D = \underline{\rm Ker}(D) - \underline{\rm Coker}(D)$$

In our framework, N is identified with $\mathcal{M}$, the moduli space of theories, and each theory $n \in N$ has a supercharge D_n. $\operatorname{Coker}(D_n)$ is then identified with the fermionic zero modes. Therefore, in supersymmetric quantum field theories with the vanishing property, $\operatorname{Coker}(D_n) = 0$ for all $n \in \mathcal{M}$ and index bundle is $\underline{\rm Ind}(D) = \underline{\rm Ker}(D)$ which is the vacuum bundle.

The Families Index Theorem suggests that the vacuum bundle exists for supersymmetric theories whenever the following conditions are satisfied: (a) a compact manifold $\mathcal{M}$ can be identified with (possibly a subset of) the Moduli space, (b) the vanishing property holds at every point $\mathcal{T} \in \mathcal{M}$, and (c) the supercharge $Q_\mathcal{T}$ is a closed, densely defined Dirac-type elliptic operator with $(Q_\mathcal{T})^2 = H_\mathcal{T}$. We give an existence proof in the next section that does not rely directly on the index theorem.

5. Construction of the Vacuum Bundle

In this section we give the ground states of the Wess-Zumino models considered above a geometrical structure, by first constructing the moduli space of admissible superpotentials (the *base space* of the vector bundle), and then proving that the ground states vary holomorphically over this space.

5.1. The Base Space. In this section we give a detailed description of the Janowsky-Weitsman moduli space, showing it to be a differentiable manifold, and therefore of suitable character to function as the base space for a vector bundle.

The polynomial superpotential is $W_\lambda(x) \equiv \lambda^{-2}\widetilde{W}(\lambda x)$, with the assumptions

(A) $\widetilde{W}'$ must have $n-1$ distinct zeros, where $n = \deg \widetilde{W}$, and
(B) $\left|\widetilde{W}''\right| = 1$ at each zero of $\widetilde{W}'$.

The first condition is motivated by the fact that the bosonic potential $|W_\lambda'(\phi)|^2$ has minima where $\widetilde{W}'$ has zeroes. Scaling $\lambda \to 0$ increases the distance between and the depth of the potential wells. Roughly speaking, the moduli space of theories we will consider is the space of potentials satisfying (A) and (B). Such potentials exist; a one-parameter family with degree $2n'$ is given for $\beta \in (0,1)$ by

$$\widetilde{W}_\beta'(z) = \prod_{k=1}^{n'-1}\left(2\sin\frac{\pi k}{n'}\right)^{-1}\prod_{k=1}^{n'}\left[\frac{(z-e^{2\pi i k/n'})(z-e^{2\pi i(k+\beta)/n'})}{2\sin\frac{\pi(k+\beta)}{n'}}\right].$$

The existence of such families suggests that the space of superpotentials is a topological space containing continuous paths. In fact, the space of potentials satisfying (A) has a very natural geometry; and the restriction (B) will be removed by a mass perturbation which we will analyze.

We let $\mathbb{C}[X]_n$ denote the set of all polynomials of degree n in one variable over $\mathbb{C}$. We let $\mathcal{Z}_{n,k}$ denote the space of all $p(X) \in \mathbb{C}[X]_n$ s.t. p has exactly k distinct zeros. Also let $P(n,k)$ denote the number of partitions of n with length k and no zero entries. For $1 < k < n$, the space $\mathcal{Z}_{n,k}$ has $P(n,k)$ distinct connected components, but for $k = n$ (the case of our interest), the polynomial is uniquely determined by the n distinct zeros, together with an overall scaling factor. Therefore,

$$\mathcal{Z}_{n,n} = \mathbb{C} \times \Big\{(z_1,\ldots,z_n) : z_i \neq z_j \ \forall i,j\Big\} \tag{19}$$

In particular, (19) shows that $\mathcal{Z}_{n,n}$ is $\mathbb{C}^{n+1}$ minus a closed set, and therefore a differentiable manifold.

In the case of the Janowsky-Weitsman space, we need to characterize the set of possible $\widetilde{W} \in \mathbb{C}[z]_n$ such that $\widetilde{W}' \in \mathcal{Z}_{n-1,n-1}$. Quite generally, if $S \subset \mathbb{C}[z]$ is a finite-dimensional manifold, we define the notation

$$\int S \equiv \{f(z) \in \mathbb{C}[z] : f'(z) \in S\}. \tag{20}$$

Then there is a bijective mapping $\int S \longleftrightarrow \mathbb{C}\times S$ given by mapping the pair $(c, g(z)) \in \mathbb{C}\times S$ to the polynomial $c + \int_0^z g(w)dw$. The space $\int S$ inherits the structure of a differentiable manifold in the natural way by declaring that this bijection is a diffeomorphism.

We conclude that condition (A) is equivalent to the statement:

$$\widetilde{W} \in \int \mathcal{Z}_{n-1,n-1}.$$

The second condition (B) is more problematic because it states that $(\forall\, i)\, \widetilde{W}''(z_i) \in S^1$, and S^1 is a closed set in $\mathbb{C}$. This problem is resolved by noting that the results of Janowsky-Weitsman are invariant under perturbations of the form

$$W_\lambda(x) = \lambda^{-2}\widetilde{W}(\lambda x) + \lambda^{-1}\epsilon w(\lambda x) \tag{21}$$

where w is also a polynomial of degree n and ϵ is a small parameter. This breaks any artificial symmetry due to the mass restriction (B). We wish to analyze this symmetry breaking and the effect on the masses in greater detail. In order to do this, we establish that adding a small perturbation to a polynomial with its zeros separated causes each mass $\widetilde{W}''(z_k)$ to be perturbed within a similarly small neighborhood of its unperturbed value. We call this *fine tuning* of the zeros.

Consider the problem of defining a function $w = f(z)$ by solution of the algebraic equation $G(w, z) = 0$ where G is an irreducible polynomial in w and z. If G is arranged in ascending powers of w, this equation can be written

$$g_0(z) + g_1(z)w + \cdots + g_m(z)w^m = 0 \tag{22}$$

If we imagine a particular value z_0 to be substituted for z, we have an equation in w which, in general, will have m distinct roots $w_0^{(1)}, w_0^{(2)}, \ldots, w_0^{(m)}$. An exception takes place if and only if

(i) $g_m(z_0) = 0$, in which case the degree of the equation is lowered, or
(ii) $G(z_0, w) = 0$ has multiple roots.

The second case can occur if and only if the discriminant, which is an entire rational function of the coefficients, vanishes. If $G(z, w)$ is irreducible, then the discriminant $D(z)$ does not vanish identically but is a polynomial of finite degree. Thus the exceptions *(i)* and *(ii)* can occur for only a finite number of special values of z, which we denote by $a_1, a_2, \ldots, a_r$, and which we call *excluded points*.

By the implicit function theorem, for any non-excluded z_0, there are n distinct function elements $\omega_1, \ldots, \omega_n$ such that

$$G(z, \omega_j(z)) = 0\,. \tag{23}$$

If we continue one of these function elements ω_j to another non-excluded point z_1, we get another function element (over z_1) that satisfies (23). In this way, the equation $G(z, w) = 0$ defines a multi-valued function, or Riemann surface; we state this as a lemma.

Lemma 2. *In the punctured plane*

$$H = \mathbb{C} \setminus \{a_1, \ldots, a_r\}$$

the equation $G(z, w) = 0$ *defines precisely one* m*-valued regular function* $w = F(z)$.

Lemma 2 and the discussion preceding it apply to the special case in which all but one of the functions $g_i(z)$, defined in eq. (22), are constant,

$$g_i(z) = \begin{cases} c_i, & i \neq k \\ z, & i = k \end{cases}, \quad c_i \in \mathbb{C}$$

Away from the excluded points $\{a_\nu\}$ associated to this choice, the zeros of $\sum_{i=0}^{m} g_i(z)w^i$ are distinct and vary as analytic functions of the coefficient of w^k. Repeating this procedure for each $k = 1 \ldots m$, we conclude that away from excluded points, the zeros depend holomorphically on each coefficient.

We now reformulate this result in a way that is relevant to quantum field theory, which we state as Theorem 6. For a polynomial $w(x) = \sum a_i x^i$, we define $\|w(x)\|^2 = \sum |a_i|^2$, which gives $\mathbb{C}[x]_n$ the topology of Euclidean space.

Theorem 6 (Fine Tuning). *Consider a fixed polynomial superpotential $\widetilde{W}(x)$. Let $\mathcal{N}$ be a neighborhood of 0 in the space $\mathbb{C}[x]_n \cup \{0\}$. Let $Z = \{\xi_1, \ldots, \xi_n\}$ be the zero set of $\widetilde{W}'(x)$, which we assume is nondegenerate, and let Z_w denote the zero set of $\frac{d}{dx}(\widetilde{W}(x) + w(x))$. For $\mathcal{N}$ sufficiently small, we assert that the union $\bigcup_{w\in\mathcal{N}} Z_w$ takes the form $\bigcup_{i=1}^n \Omega_i$ where for each i, Ω_i is an open neighborhood of ξ_i and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $\max_i |\Omega_i| < \epsilon$ whenever $|\mathcal{N}| < \delta$ (an absolute value sign denotes the diameter in the natural metric).*

This analysis shows that a differentiable manifold of potentials which allow for the convergence of cluster expansions is given by the integral, in the sense of (20), of the set of all degree $n-1$ polynomials f with all zeros ξ_i distinct, and such that $f'(\xi_i) \in \Omega_i$ for all i, where Ω_i are nonoverlapping open sets. We denote this manifold by $\mathcal{JW}$.

5.2. The Fibers of The Vacuum Bundle. The following theorem is an analytic statement about the variation of $\ker(H)$ as we change the base point in the manifold of coupling constants. As the vectors in $\ker(H)$ are identified with physical ground states (also called *vacua*), Theorem 7, together with our characterization of the moduli space $\mathcal{JW}$ of admissible potentials, implies the existence of a vector bundle built from the vacua, as predicted by Cecotti and Vafa [2]. We propose that results of this type be termed *vacuum bundle estimates*.

Theorem 7. *Let M be a complex manifold and let $W : M \times \mathbb{C}^n \to \mathbb{C}$ be a function which is holomorphic in its dependence on $m \in M$ and in its dependence on $z \in \mathbb{C}^n$. Assume that $W(m,z)$ is polynomial in the z variable with $\deg W$ equal to a constant function on M, and for each $m \in M$, the $N = 2$ Wess-Zumino Hamiltonian H_W defined by choosing $W(m,z)$ as polynomial superpotential satisfies $n_-(H_W) = 0$. Let $\mathcal{V}(m)$ denote the ground state subspace of the Wess-Zumino model defined by $W(m,z)$, i.e. $\mathcal{V}(m) = \ker(H_{W(m,z)})$. Then $\mathcal{V}$ is a holomorphic vector bundle over M.*

Proof of Theorem 7. We wish to show holomorphicity of the ground state vector space; by the vanishing property ($n_- = 0$), we may restrict our considerations to bosonic ground states. Write $\mathcal{H}_b = \mathcal{H}_\leq \otimes \mathcal{H}_>$ where $\mathcal{H}_\leq$ is spanned by states of the form $R\Omega_0$, with R ranging over all finite polynomials in creation operators $a_j^*(p)$ for $|p| \leq (n-1)\kappa$. Our strategy is to first show the desired result for a theory with an infrared cutoff, and then show that the desired property is preserved in the limit as the cutoff is removed. We refer to M as *theory space*.

The bosonic, cutoff Hamiltonian takes the form

$$H_{m,b}(\kappa) = H_m{}^{\leq} \otimes I + I \otimes H_0{}^{>} \tag{24}$$

where $H_0{}^{>}$ contains no interacting modes, and $H_m{}^{\leq}$ is equivalent to a Schrödinger operator $-\triangle + V_m$ acting on $L^2(\mathbb{R}^n)$ with polynomial potential V_m. Here, κ is the momentum space cutoff.

As $m \in M$ changes holomorphically, it follows from well-known results of constructive field theory (see for example Arthur Jaffe's PhD thesis) that the Schrödinger operators $-\triangle + V_m$ meet the conditions of Theorem 5. We conclude that each of the operators appearing in eq. (24) depends holomorphically on the parameters m in theory space. Since none of our results depend on the global geometry or topology of M, we are free to choose, once and for all, a point p in M and a set $\chi = (z_1, \ldots, z_n)$ of complex coordinates near p. We choose the origin of the coordinate system so that $\chi = 0$ in $\mathbb{C}^n$ corresponds to $p \in M$, and prove that the relevant operators are holomorphic in χ at $\chi = 0$.

We may conclude that $H_{\chi,b}(\kappa)$ is holomorphic in the complex parameter χ, in the generalized sense for unbounded operators. This implies that the cutoff resolvent $R(\kappa, \chi, \zeta) = (H(\kappa, \chi) - \zeta)^{-1}$ is bounded-holomorphic in χ. Jaffe, Weitsman, and Lesniewski have shown that the cutoff resolvent is norm continuous in κ and moreover

$$\lim_{\kappa \to \infty} (H(\kappa, \chi) - \zeta)^{-1} = (H(\chi) - \zeta)^{-1}$$

We need to show that the norm limit $R(\chi, \zeta)$ is also bounded-analytic in χ. One way to see this is to show that the derivative with respect to χ of the cutoff resolvents converges, in the limit as the cutoff is removed, to the derivative of $(H(\chi) - \zeta)^{-1}$.

Since $H(\kappa, \chi)$ is the perturbed cutoff Hamiltonian, we have

$$\frac{\partial}{\partial \chi}(H(\kappa, \chi) + \zeta)^{-1} = \frac{1}{2\pi i} \oint_C (H(\kappa, \chi') + \zeta)^{-1} (\chi' - \chi)^{-2} d\chi' \tag{25}$$

where C is a circle in the complex χ-plane around the point of holomorphicity (in this case $\chi = 0$). The limit of the derivative of the resolvent as $\kappa \to \infty$ is the limit of the l.h.s. of (25), which must equal the limit of the r.h.s. Since C is compact, the integrand is uniformly continuous, and hence the $\kappa \to \infty$ limit can be interchanged with $\oint_C$. Moving the limit inside, we use the fact that the resolvents $(H(\kappa, \chi) + \zeta)^{-1}$ converge in norm to the resolvent of the limiting theory $(H(\chi) + \zeta)^{-1}$. So the limit of the derivative of the resolvent as $\kappa \to \infty$ is

$$\lim_{\kappa \to \infty} \left(\frac{\partial}{\partial \chi}(H(\kappa, \chi) + \zeta)^{-1} \right) = \frac{1}{2\pi i} \int_C (H(\chi') + \zeta)^{-1} (\chi' - \chi)^{-2} d\chi'$$

which equals the derivative of the resolvent of $H(\chi)$. It now follows by Theorem 1 that the Hamiltonian of the limiting theory is holomorphic in χ.

The Hamiltonian $H(\chi = 0)$ has a spectral gap above the ground state eigenvalue (in fact it is essentially self-adjoint with trace class heat kernel, so the spectrum consists entirely of isolated points). Therefore we can apply Lemma 2: specifically, we choose the rectifiable Jordan curve required by the Lemma to be a circle enclosing only the ground state eigenvalue. In the notation of Lemma 2, the vacuum states are basis vectors for the subspace $M'(\chi)$ and Lemma 2 implies that $M'(\chi)$ is holomorphic in a neighborhood of $\chi = 0$. This completes the proof. □

6. The tt^* Connection

In this section we present a rigorous construction of a connection on the vacuum bundle. The connection which we construct was originally discovered in a physics context by S. Cecotti and C. Vafa [2]. This is a generalization to $N = 2$ Wess-Zumino field theory of the representation of Berry's geometrical phase in ordinary quantum mechanics as the holonomy of a connection on a principal $U(1)$ bundle.

The WZ Hamiltonian in the limit as the cutoff is removed is well defined on the tensor product $\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}_f$ (this is the main theorem of [21]). The result holds for a wide class of superpotentials, thus the fixed Hilbert space $\mathcal{H}$ that will be necessary to define the tt^* connection exists. The result on the existence of the vacuum bundle shows that there is indeed a subspace $V(m)$ in this fixed Hilbert space $\mathcal{H}$ for each m in the parameter space $\mathcal{M}$ of superpotentials.

A *covariant derivative* on a vector bundle $E \to M$ is a differential operator

$$\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$$

satisfying the Leibniz rule: if $s \in \Gamma(M, E)$ and $f \in C^\infty(M)$ then $\nabla(f \cdot s) = df \otimes s + f\nabla s$. A covariant derivative so defined automatically extends to give a map

$$\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E).$$

Consider a coordinate chart $U \subset \mathcal{M}$ with local coordinates (x^a), $a = 1 \dots n$. Let $V \to \mathcal{M}$ be the vacuum bundle. The restriction $s|_U$ of a section $s \in \Gamma(\mathcal{M}, V)$ can be identified via the coordinates (x^a) with a function on $\mathbb{R}^n$ taking values in $\mathcal{H}$, which we denote by $s(x^1, \dots, x^n)$. We write $\partial_a s$ for the partial derivative $\partial s(x^1, \dots, x^n)/\partial x^a$.

Suppose that the states $|\alpha(x)_i\rangle,\ i = 1, \dots, \mathrm{rank}(V)$ form an ON basis of $V(x)$ for each $x \in U$, and vary smoothly in their dependence on x. Equivalently, the $|\alpha(x)_i\rangle$ form a local orthonormal frame for V. Consider a curve $\lambda \to x_\lambda$ mapping $(0, 1)$ into U. We note that in the difference quotient

$$\frac{d}{d\lambda}\Big|_{\lambda=0} |\alpha(x_\lambda)\rangle = \lim_{h\to 0} \frac{1}{h}\Big(|\alpha(x_{\lambda+h})\rangle - |\alpha(x_\lambda)\rangle\Big),$$

$|\alpha(x_{\lambda+h})\rangle$ and $|\alpha(x_\lambda)\rangle$ represent vacuum states of different Hamiltonians, and hence the difference $|\alpha(x_{\lambda+h})\rangle - |\alpha(x_\lambda)\rangle$ is not a ground state, and even if the spaces $V(x)$ are closed, the partial derivative $\partial_a s$ of a section s can lie outside of V.

We define a covariant derivative on V by the equation

$$(\nabla s)_m \equiv P_{V(m)}(\partial_a s)_m \, dx^a$$

so that $\nabla s \in \Gamma(M, T^*M \otimes V)$. $P_{V(m)}$ denotes the projection onto the vacuum subspace $V(m) \subset \mathcal{H}$. A sum over each index appearing in both upper and lower positions is implied. Thus ∇s is a one-form with coefficients in V, i.e. a section of $\Omega^1(M) \otimes V$.

Since the states $|\alpha(x)_j\rangle$ are locally a basis of V, we can determine the matrix for ∇ in this basis:

$$\nabla|\alpha_i\rangle = |\alpha_j\rangle \omega^j{}_i$$

where $\omega = (\omega^j{}_i)$ is a matrix-valued one-form. By the definition of ∇,

$$P_V \partial_a |\alpha_i\rangle \, dx^a = |\alpha_j\rangle \omega^j{}_i$$

Taking the inner product with $\langle\alpha_k|$ yields an expression for the connection forms ω_{ki}

$$\langle\alpha(x)_k| P_{V(x)} \frac{\partial}{\partial x^a} |\alpha(x)_j\rangle = \langle\alpha(x)_k|\alpha_j\rangle \omega^j{}_i = \omega_{ki}$$

We now show that for the purposes of computing the connection forms, it is not necessary to insert the projection operator P_V. Since the states $|\alpha(x)_j\rangle$ are a local frame for V, we can write

$$P_V = \sum_j |\alpha(x)_j\rangle\langle\alpha(x)_j|$$

It follows that

$$\begin{aligned}\langle\alpha(x)_k|P_{V(x)}\frac{\partial}{\partial x^a}|\alpha(x)_j\rangle &= \langle\alpha(x)_k|\sum_j|\alpha(x)_j\rangle\langle\alpha(x)_j|\frac{\partial}{\partial x^a}|\alpha(x)_j\rangle\\ &= \sum_j\delta_{jk}\langle\alpha(x)_j|\frac{\partial}{\partial x^a}|\alpha(x)_j\rangle\\ &= \langle\alpha(x)_k|\frac{\partial}{\partial x^a}|\alpha(x)_j\rangle\end{aligned}$$

These considerations do not depend in an essential way on the intended application to $(2,2)$ supersymmetric QFT's. The above discussion in fact proves the following general existence theorem:

Theorem 8. *Let $\mathcal{V} : M \to Gr_k(\mathcal{H})$ be a smooth map from M into the Grassmannian of k-dimensional closed subspaces of a fixed Hilbert space $\mathcal{H}$. Then under a suitable local condition on the transition functions, the association $x \to \mathcal{V}(x)$ gives rise to a C^∞ vector bundle $E \xrightarrow{\pi} M$, where $E = \bigcup_{x\in M}\mathcal{V}(x)$. This bundle inherits a natural Hermitian structure g from the Hilbert space inner product, defined by $g_x(\phi,\psi) = \langle\phi|\psi\rangle$, where $\phi,\psi \in E_x$. The Levi-civita connection corresponding to this Hermitian structure is given explicitly by the formula*

$$(\nabla s)_m \equiv P_{V(m)}(\partial_a s)_m\, dx^a \quad \textit{for} \quad s \in \Gamma(E)$$

In a specific choice of a local orthonormal frame, the connection forms ω_{ki} are given by

$$\omega_{ki} = \Big\langle\alpha(x)_k\Big|\frac{\partial}{\partial x^a}\Big|\alpha(x)_j\Big\rangle dx^a$$

6.1. Application: The CFIV Index. The ground state metric arises in calculations of the CFIV index [6], as well as in other important calculations. The infinite volume theory entails degenerate vacua at +/- spatial infinity, and what is actually well defined is the trace $\mathrm{Tr}_{(a,b)}$ over the (a,b) sector, where a and b are indices which label the different ground states. Physicists calculate [6] that for a cylinder of length L and radius β, the CFIV index $Q_{ab} \equiv i\beta L^{-1}\,\mathrm{Tr}_{(a,b)}(-1)^F F e^{-\beta H}$ is given by

$$Q_{ab} = -(\beta\, g\partial_\beta g^{-1} + n)_{ab} \tag{26}$$

where n is the number of fields in the Landau-Ginzburg theory and g is the ground state metric. Thus the calculation of the CFIV index in the (a,b) sector is reduced to calculating the metric g. In principle this is done by integrating the tt^* differential equation which g satisfies, however these equations are complicated. One simplification is to transform to a special gauge in which the tt^* equation becomes

$$\overline{\partial}_j(g\partial_i g^{-1}) = \beta^2[C_i, gC_j^\dagger g^{-1}] \tag{27}$$

where C_{ij}^k is the structural tensor for the chiral ring.

Eq. (27) is an $N \times N$ matrix of differential equations involving the components of g, where N is the number of ground states, or chiral fields. These equations are integrable, and in certain cases equivalent to classical equations of mathematical physics, which are generally Toda systems. Therefore (27) determines the ground state metric non-perturbatively. Using the resulting solution in (26) gives the CFIV index. Other tt^* equations include a flatness condition for the connection, $[D_i, D_j] = 0$ and the integrability condition for the tensor C_{ij}^k, i.e. $D_i C_{jk}^\ell = D_j C_{ik}^\ell$.

Results of this paper show that the structures (vacuum bundle, metric g) used in the above heuristic argument do exist. Thus our results are basic for any rigorous study of the CFIV invariant in infinite volume.

7. Holomorphic Quantum Mechanics

We describe a model of $N = 2$ quantum mechanics with interactions parameterized by a holomorphic superpotential $W(z)$. The coupling constant space is usually taken to be $\mathbb{C}^{n+1}$ (a vector in $\mathbb{C}^n$ corresponds to a coefficient vector for a polynomial W of degree n), although many of the results generalize to the situation in which we replace $\mathbb{C}^{n+1}$ by an arbitrary Stein manifold [19]. For this reason the model is also called *holomorphic quantum mechanics*.

The Hamiltonian is a mathematically well-defined generalization of the Hamiltonians of various phenomenological systems. Application of this model to a system of interacting pions is described in [20]. We prove that the vector space of ground states varies continuously in the Hilbert Grassmannian, under suitable perturbations. This is a special case of the fundamental vacuum bundle estimate which was introduced as Theorem 7, however the $N = 2$ quantum mechanics model is sufficiently simple that it is possible to understand the vacuum bundle estimate in an elementary way.

The model we will study is the one-dimensional version of $N = 2$ supersymmetric Landau-Ginzburg quantum field theory. In this model, $z(t)$ denotes one bosonic degree of freedom, and ψ_1, ψ_2 are fermionic degrees of freedom. The Lagrangian

$$\mathcal{L} = |\dot{z}|^2 + i(\overline{\psi_1}\dot{\psi}_2 + \overline{\psi_2}\dot{\psi}_1) + \overline{\psi_1}\psi_1 \partial^2 V + \overline{\psi_2}\psi_2 (\partial^2 V)^* - |\partial V|^2$$

is parameterized by $V(z)$, a holomorphic polynomial of degree n in z. In supersymmetric models, the Hamiltonian may be expressed as the square of a supercharge. The latter is computed from the supersymmetry transformations and the Noether theorem. The result of that calculation gives:

$$H = Q^2 = -\partial\overline{\partial} - \overline{\psi_1}\psi_1 \partial^2 V - \overline{\psi_2}\psi_2 (\partial^2 V)^* + |\partial V|^2$$

This is motivated by the application to a quantum theory with $N = (2, 2)$ supersymmetry, in which we study the space of ground states:

$$V = \left\{ |\,\alpha\,\rangle \in \mathcal{H} : Q|\,\alpha\,\rangle = Q^\dagger |\,\alpha\,\rangle = 0 \right\}$$

We define a map $\mathcal{V} : \mathcal{M} \to Gr(\mathcal{H})$, i.e. from the moduli space $\mathcal{M}$ of admissible supersymmetric quantum theories into the Hilbert Grassmannian of $\mathcal{H}$, called the *vacuum*:

$$m \xmapsto{\mathcal{V}} \ker H(m)$$

In order to define the vacuum map explicitly, we first review the results of [20]. Every zero mode arises from a pair (f, g) of $L^2(\mathbb{C})$ functions, where g satisfies the differential equation

$$(-\overline{\partial}\partial + |\partial V|^2)g + (\partial^2 V/\partial V)^* \partial g = 0 \tag{28}$$

and f satisfies the complex conjugate equation. We refer to (28) as the *supercharge-kernel equation*. For $V = \lambda z^n$, (28) becomes

$$-\partial\overline{\partial}g + (n-1)\overline{z}^{-1}\partial g + \left|n\lambda z^{n-1}\right|^2 g = 0 \tag{29}$$

Representing z in polar coordinates (r, θ) and writing $g(r, \theta)$ as a Fourier series in the angular variable

$$g(r, \theta) = \sum_{m\in\mathbb{Z}} u_m(r)e^{im\theta}$$

yields an ODE for the radial functions:

$$-u_m'' + \frac{2n-3}{r}u_m' + \left(4n^2\lambda^2 r^{2n-2} + \frac{m(m-2n+2)}{r^2}\right)u_m = 0 \tag{30}$$

This equation takes the general form (31); we study regularity of such objects in Lemma 3.

Lemma 3. *Solutions of equations of the type*

$$u'' + Ar^{-1}u' + \left(B\lambda^2 r^\alpha + Cr^{-2}\right)u = 0 \tag{31}$$

display regularity in the parameter λ, where $A, B,$ and C are nonzero real constants.

Proof of Lemma 3. A generic second-order initial value problem of the form (31) can be transformed into a system of equations of first order. Such systems are equivalent to vector integral equations of Volterra type

$$\mathbf{y}(x;\lambda) = \mathbf{g}(x;\lambda) + \int_{\alpha(\lambda)}^{x} \mathbf{k}(x, t, \mathbf{y}(t;\lambda);\lambda)\,dt. \tag{32}$$

Here x and t are always real, but $\mathbf{g}, \mathbf{k}$, and $\mathbf{y}$ may be complex-valued. More than one real or complex parameter is allowed, i.e. $\lambda \in \mathbb{R}^m$ or $\mathbb{C}^m$. Theorem 13.III in [16] shows that the solution $\mathbf{y}$ to an equation of the form (32) is holomorphic in the parameter λ. □

Lemma 4. *Let $f_1, \ldots, f_n$ be continuous maps from a topological space Λ into a Hilbert space $\mathcal{H}$ such that $V(\lambda) := \mathrm{Span}\{f_1(\lambda), \ldots, f_n(\lambda)\}$ is n-dimensional for any λ. Then $\lambda \longmapsto V(\lambda)$ is a continuous map into $Gr(\mathcal{H})$. Moreover, if Λ is a complex manifold and each f_j is holomorphic, then so is $V(\lambda)$.*

Proof of Lemma 4. For each $\phi \in \mathcal{H}$, let $N_\phi(A) = \|A\phi\|$. The collection $\{N_\phi \mid \phi \in \mathcal{H}\}$ is a separating family of seminorms on $\mathcal{B}(\mathcal{H})$, and the associated topology is the strong operator topology. Now suppose $t \to \psi(t)$ is a continuous map from Λ to the unit ball of $\mathcal{H}$. Then the projector onto the ray containing $\psi(t)$ is $P_{\psi(t)} = |\,\psi(t)\,\rangle\langle\,\psi(t)\,|$, and $\left\|P_{\psi(t)}\right\| = |\langle\,\psi(t)\,|\,\phi\,\rangle|$, which is continuous in t; thus the Lemma is proved for $n = 1$. In case $n = \dim V(t) > 1$, we have $\left\|P_{V(t)}\phi\right\| \le \sum_{i=1}^{n}\|P_{\psi_i}\phi\| = \sum_{i=1}^{n}|\langle\,\psi_i(t)\,|\,\phi\,\rangle|$, and the desired result follows by an "ε/n argument." The proof of holomorphicity is similar. □

Lemma 3 and Lemma 4 together imply the following

Theorem 9. *The vector space of vacuum states of the $N = 2$ Landau-Ginzburg model of quantum mechanics varies holomorphically in the Hilbert Grassmannian over a moduli space of coupling parameters diffeomorphic to $\mathbb{C}^n \times (\mathbb{C} - \{0\})$, and determines a vector bundle of rank $(n-1)$.*

Proof of Theorem 9. We can write down the zero modes as explicit functions, and thus there are $n-1$ linearly independent zero modes if $n = \deg V$. Let $\mathbb{C}[z]_n$ denote the space of polynomials with complex coefficients of degree exactly n. Then $\mathbb{C}[z]_n$ is the space of $\sum_{k=0}^{n} a_k z^k$ such that $a_n \neq 0$, and is therefore isomorphic to the open submanifold $\mathbb{C}^n \times (\mathbb{C} - \{0\})$ of $\mathbb{C}^{n+1}$. By Lemma 3, each of the $n-1$ linearly independent zero modes is holomorphic as a function of the parameters $(a_0, \ldots, a_k) \in \mathbb{C}^n \times (\mathbb{C} - \{0\})$. □

8. Directions for Further Research

Let the coupling constant space of a family of Wess-Zumino models be $\mathcal{M}$, and let the vacuum bundle be $\mathcal{V} \to \mathcal{M}$. The ground state metric $g_{i\bar{j}}$ is a Hermitian metric on $\mathcal{V}$, and therefore it defines a geodesic flow on $\mathcal{M}$ in situations when the vacuum bundle can be identified with the tangent bundle $T\mathcal{M}$. Renormalization also gives a flow on the moduli space $\mathcal{M}$ of theories, but in this case there is a preferred vector field $\boldsymbol{\beta}$ which serves as the dynamical vector field of the flow, known as the beta function.

In a Euclidean quantum field theory defined by an action $S(g,a) = \int \sigma(g,a,x)dx$ where $g = (g^1, g^2, \ldots)$ is a set of coupling constants and a is a UV cutoff, we assume there exists a one-parameter semi-group R_t of diffeomorphisms on $\mathcal{M}$ such that the theory $S(R_t g, e^t a)$ is equivalent to the theory $S(g,a)$ in the sense of correlators being equal at scales $x \gg e^t a$. The β function is defined by $dg^i = \beta^i(g)dt$, thus the vector field $\boldsymbol{\beta}$ generates the flow.

Zamolodchikov defined a metric G_{ij} on $\mathcal{M}$ which schematically takes the form

$$G_{ij} = x^4 \langle \Phi_i(x) \Phi_j(0) \rangle \big|_{x^2 = x_0^2} \quad \text{where} \quad \Phi_i(x) = \frac{\partial}{\partial g^i} \sigma(g,a,x)\,.$$

Up to singularities, the flow lines determined by acting on a single point $g \in \mathcal{M}$ with R_t for all $t \in \mathbb{R}$ coincide with geodesics of G_{ij}.

It would be of fundamental importance to develop a mathematically rigorous version of the renormalization group for the constructive Wess-Zumino model considered in this paper, and then in those cases when the ground state metric $g_{i\bar{j}}$ computes lengths of vectors in the tangent bundle $T\mathcal{M}$, to prove an exact relationship between the ground state metric $g_{i\bar{j}}$ and Zamolodchikov's metric G_{ij}.

A second important unsolved problem is to determine the largest possible moduli space for two-dimensional $N = 2$ Wess-Zumino theories in which the vanishing property holds. The cluster expansion is one of the most refined estimates known for stability of such theories, and yet the cluster expansion is certainly weaker than the optimal bound. For these reasons, we expect that the moduli space we have used in this paper is a submanifold of the optimal moduli space for the vacuum bundle.

A new research direction in functional analysis is suggested following Theorem 5. Moreover, it is likely that additional new mathematics would be found in a further exploration of the interplay between the geometry of the vacuum bundle and the infinite-dimensional analysis of constructive quantum field theory.

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