Planning a sequence of trades extending into the future is a very common problem in finance. All trading is costly, and the need for intertemporal optimisation is more acute when trading costs are considered. The total cost due to market impact is known to be superlinear as a function of the trade size (Almgren et al (2005) measured an exponent of about 0.6 for impact itself, hence 1.6 for total cost), implying a large order may be more efficiently executed as a sequence of small orders. Indeed, optimal liquidation paths had already been studied by Almgren & Chriss (1999) under an idealised linear impact model, leading to quadratic total cost.

A similar, but more complex, problem is faced by the discretionary trader, who can set the time horizon and wait to deploy an alpha strategy until there is a trading path with favourable expected utility. Further, the drivers of demand for trading may differ vastly at different horizons. Disagreement among alpha models defined at various horizons is, in fact, commonplace in quantitative trading. Gârleanu & Pedersen (2013) studied the multiperiod quantitative-trading problem under the somewhat restrictive assumptions that the alpha models follow mean-reverting dynamics and that the only sources of trading frictions are purely linear market impacts (leading to purely quadratic impact-related trading costs).

A third problem, related to the first two, is the practicality of hedging derivatives contracts when the trading cost of dynamic offsetting replicating portfolios is taken into account. This problem is routinely faced by the office of the chief investment officer at an investment bank, which must balance risk with the cost of trading a large hedging position.

In this paper we present a general framework that encompasses all of these types of problems and establishes an intuitively appealing link to the theory of Bayesian statistics. Intuition is most valuable when it is also useful, however, and perhaps the best feature of our framework is that intuition leads to a straightforward algorithm for solving the problem. This algorithm applies to the realistic case when we assume the investor’s utility function can be approximated by the first two terms in its Taylor series. Thus the problem we treat initially is that of maximising $u(x)$ where:

$$u(x) := E[\pi(x)] - (\gamma/2)\sqrt{\pi(x)}$$

where $\gamma > 0$ is the risk-aversion parameter. Just after Intuition 1 below, we discuss how our framework can be applied to more general problems that transcend (2).

We often refer to a planned portfolio sequence $x = (x_1, x_2, \ldots, x_T)$ simply as a ‘path’. Similarly, we sometimes refer to (2) as the ‘utility of the path $x$’, while remembering the more complex link to utility theory noted above. Our task, in this simpler language, is to find the maximum-utility path $x^* = \arg\max_x u(x)$.

Combining (1) with (2), and defining $\alpha_t := E[r_{t+1}]$ and $\Sigma_t := \sqrt{\text{Var}[r_{t+1}]}$, we have:

$$u(x) = \sum_t \left[ x_t^T \alpha_t - \frac{\gamma}{2} x_t^T \Sigma_t x_t - c_t(x_{t-1}, x_t) \right]$$

Neglecting terms that do not depend on $x$, the first two terms of (3) are (up to a sign) equivalent to:

$$b_{Y, \Sigma_t}(x_t, y_t) := \frac{1}{2} (y_t - x_t)^T \gamma \Sigma_t (y_t - x_t)$$

where:

$$y_t := (\gamma \Sigma_t)^{-1} \alpha_t$$

The latter is a classic mean-variance portfolio, which is known to be the solution to a myopic problem without costs or constraints, and $b_{Y, \Sigma_t}$ measures variance of the tracking error. Then up to $x$-independent terms:

$$u(x) = -\sum_t [b_{Y, \Sigma_t}(x_t, y_t) + c_t(x_{t-1}, x_t)]$$

In any multiperiod optimisation problem with transaction costs, one can always ask what the solution would be in an ideal world without transaction costs or, equivalently, at the limit as costs tend to zero. We call this solution the ideal sequence, and always denote it by $y_t$.

**INTUITION 1** Multiperiod portfolio optimisation is mathematically equivalent to optimally tracking a sequence $y_t$, called the ideal sequence, which is the portfolio sequence that would be optimal in a transaction-cost–free world.
The general guiding principle, expressed as Intuition 1, extends beyond the case in which the ideal sequence is \((y \Sigma t)^{-1} a_t\) and, indeed, beyond the case in which \(y_t\) has a clean derivation from a utility function. For computing optimal liquidation paths in the spirit of Almgren & Chriss (1999), the ideal sequence is clearly \(y_t = 0\) for all \(t\). For hedging exposure to derivatives, \(y_t\) should be our expectation of the offsetting replicating portfolio at all future times until expiration. Tracking the portfolios of Black & Litterman (1992) is also a special weighted by (7). We will refer to this agent as the random trader.

For hedging exposure to derivatives, \(y_t\) is the solution to a mean-variance problem with a Bayesian posterior distribution for the expected returns. Since the posterior is Gaussian in the original Black-Litterman model, the two-moment approximation to utility is exact, and one simply replaces \(a_t\) and \(\Sigma_t\) with the appropriate quantities.

In practice, investors’ utility functions may depend on higher moments (skewness, kurtosis) or partial moments of the distribution of final wealth (Zakamouline & Koekebakker 2009). Optimal portfolio selection in this case is a hard problem, even without transaction costs, as the objective function need not be convex. Suppose, however, that an investor with such a utility function has decided on the trading path \(y = (y_t)\) to follow in an ideal world without trading costs. Presumably this has been accomplished via solving a difficult four-moment or mean-conditional-value-at-risk optimisation problem. Once \(y\) has been determined, we may proceed with the maximisation of (6) for the purpose of tracking \(y\) in a cost-efficient manner. As we shall see, the quadratic distance function \(b_y \Sigma_y\) in (6) may be replaced with any smooth, convex function of its arguments and the optimisation technology we discuss later will still apply.

For our next piece of intuition, consider the random process model for \(x = (x_1, x_2, \ldots, x_T)\) given by:

\[
p(x) = \frac{1}{Z} \exp(\kappa u(x)), \quad Z := \int \exp(\kappa u(x)) \, dx \tag{7}
\]

for an arbitrary constant \(\kappa > 0\). In realistic models, \(Z\) is always finite. Optimising expected utility is then equivalent to predicting the most likely action of a randomly acting agent whose actions are probability-weighted by (7). We will refer to this agent as the random trader.

If \(u(x)\) is of the form (6) for any ideal sequence \(y_t\), then \(p(x)\) naturally has a product form:

\[
p(x) = \prod_t p(y_t | x_t) p(x_t | x_{t-1}) \tag{8}
\]

where:

\[
p(y_t | x_t) \propto \exp(-\kappa b_y \Sigma_y (x_t, y_t)) \quad p(x_t | x_{t-1}) \propto \exp(-\kappa c_y (x_{t-1}, x_t)) \tag{9}
\]

In fact, (8) should remind us of random process models we have seen before in other contexts.

**Intuition 2** The process model of the random trader is a hidden Markov model (HMM). The optimal trading path is the most likely sequence of hidden states, conditional on the ideal path \(y = (y_1, \ldots, y_T)\).

An HMM is based on a pair of coupled stochastic processes \((X_t, Y_t)\) in which \(X_t\) is Markov and is never observed directly. Information about \(X_t\) can only be inferred by means of \(Y_t\), which is observable and ‘contemporaneously coupled’, meaning that \(Y_t\) is coupled to \(X_t\) but not to \(X_s\) for \(s \neq t\). This coupling has a stochastic component and the conditional probability \(p(Y_t | X_t)\) is known to us, along with the transition probability \(p(X_t | X_{t-1})\) of the hidden process. These two types of terms will turn out to be exactly what we need to model the multiperiod portfolio problem.

In a given optimisation problem, trading paths \(x\) of the random trader will be modelled as realisations of the Markov process \(X_t\), and the ideal sequence \(y = (y_t)\) will correspond to a realisation of the observable \(Y_t\). The random trader’s process model, called simply \(p(x)\) above, is actually the density of \(x\) conditional on the ideal sequence \(y\), denoted by \(p(x | y)\). The most likely realisation of \(X_t\) conditional on \(y\) is the true optimal sequence in the presence of trading costs.

The Markov property and the assumption that \(Y_t\) has only contemporaneous coupling to \(X_t\) together imply:

\[
p(x | y) = \prod_t p(y_t | x_t) p(x_t | x_{t-1}) \tag{11}
\]

Any factorisation of a joint density can be represented graphically in a way that highlights conditional dependence relations. From each variable, we draw arrows to any other variables conditioned on that variable in the given factorisation. The graph for (11) is:

\[
\begin{array}{c}
y_t \downarrow \quad y_{t+1} \\
\vdots \quad \vdots \\
x_t \downarrow \quad x_{t+1} \quad \ldots
\end{array}
\tag{12}
\]

Such graphical models are often referred to as Bayesian networks. Taking logs, (11) becomes:

\[
\log p(x | y) = \sum_t [\log p(y_t | x_t) + \log p(x_t | x_{t-1})] \tag{13}
\]

Logical reasoning about the structure of the terms in (13) reveals the economic aspects of the utility function to which they must correspond. The term \(\log p(x_t | x_{t-1})\) is the only term that couples \(x_t\) with its predecessor, \(x_{t-1}\), so this term must account for all trading frictions. In other words, up to the normalisation constant that makes \(p(x_t | x_{t-1})\) a density:

\[
- \log p(x_t | x_{t-1}) = c_t(x_{t-1}, x_t) \tag{14}
\]

Similarly, \(\log p(y_t | x_t)\) is the only term that couples \(y_t\) and \(x_t\), and so it must model the utility from ‘closeness’ or ‘proximity’ to \(y\). Since this term only concerns a single moment in time, it could not possibly model anything related to portfolio transitions. Defining \(b(x_t, y_t)\) as the total disutility related to not tracking \(y_t\) exactly, we are led to:

\[
- \log p(y_t | x_t) = b(x_t, y_t) \tag{15}
\]

Relations (14)–(15) mirror the relations seen above in (9)–(10), but they hold in greater generality as they need not come from an explicit utility function.

We conclude this section by showing how our framework handles two important extensions: statistical uncertainty in parameter estimation and portfolio constraints.
The parameter estimates (or distributional inferences) that go into return forecasts, \( \alpha_t \), and risk forecasts, \( \Sigma_t \), are subject to estimation error. Out-of-sample variance depends on the precision of parameter estimates. Fortunately, this type of variance is easily handled by standard Bayesian methods. One must compute \( (2) \) with respect to a different probability measure – ie, compute \( \mathbb{E}_p[\pi(x)] - (\gamma/2)\mathbb{V}_p[\pi(x)] \) – where the mean and the variance must use the posterior predictive density \( \hat{p} \) for returns. Letting \( \theta_t \) denote the full collection of all parameters in our model for \( r_t \), and letting \( p_t(\theta_t) \) denote the posterior density of \( \theta_t \) in our Bayesian model after all data has been assimilated, the predictive density for \( r_t \) is

\[
\hat{p}_t(r_t) := \int p_t(r_t | \theta_t) p_t(\theta_t) d\theta_t
\]

and the mean-variance investor must calculate \( \mathbb{E}_\hat{p} \) and \( \mathbb{V}_\hat{p} \) using \( r_t \sim \hat{p}_t(r_t) \).

A strength of the probabilistic framework is its conceptual handling of constraints.

**Intuition 3** Constraints are regions of path space with zero probability.

For example, a long-only constraint simply means \( p(x) = 0 \) if the path \( x \) contains short positions. Practically, this means sampling from \( p \) will never generate sample paths that are infeasible with respect to the constraints and that the global maximum of \( p \) is always a feasible path, if one exists.

**Finding optimal paths: reduction of the multi-asset case to the single-asset case**

We consider the general case of \( N \) assets, \( N > 1 \), and show this problem can be reduced to iteratively optimising single-asset paths; a fact which greatly simplifies the computation and does not seem to be widely known. We then solve the single-asset case in the next section.

We will make the fairly weak assumption that our notion of distance \( b(x_t, y_t) \) from the ideal sequence \( y_t \) is a function that is convex and differentiable. These conditions are satisfied by the positive-definite quadratic form:

\[
(y_t - x_t)^T (\gamma \Sigma_t)(y_t - x_t) =: b_{\gamma \Sigma_t}(x_t, y_t)
\]

(16)

considered above and many other distance functions.

We assume trading costs can be separated into a sum of differentiable and non-differentiable terms. Total cost due to market impact in the asset being traded has the form \( |\delta|^{1+\beta} \), where \( \beta \approx 0.6 \) (Almgren et al 2005), and is thus differentiable. Hence it is also reasonable to assume any cross-asset impact is also a differentiable term. G"areanu & Pedersen (2013) in their equation (3) propose \( c(\delta) \propto \delta^T A \delta \), where \( A \) is a positive-definite matrix; this is, of course, differentiable.

We assume the non-differentiable part of trading cost is separable in the sense that it is additive over assets:

\[
c_i(x_{t-1}, x_t) = \sum_i c_i^i(x_{t-1}^i, x_t^i)
\]

(17)

where the superscript \( i \) always refers to the \( i \)th asset. For some types of costs, such as commission or borrow costs, separation (17) is true by construction. For spread pay, (17) is also reasonable; crossing the spread to execute at the inside market in one asset should not necessarily have an impact on the price of any other assets.

If the differentiable term (16) were also separable, we could optimise each asset’s trading path independently without considering the others, but we cannot: the differentiable term is usually not separable. Intuitively, trading in any one asset could either increase or decrease the tracking error variance, depending on the positions in the other assets.

Since \( x = (x_1, \ldots, x_T) \) denotes a trading path for all assets, let \( x_i^i = (x_1^i, \ldots, x_T^i) \) denote the projection of this path on to the \( i \)th asset. Let \( c_i(x_t^i) \) denote the total cost of the \( i \)th asset’s trading path. We require that each \( c_i \) be a convex function on the \( T \)-dimensional space of trading paths for the \( i \)th asset. Putting all of this together, we want to minimise \( f(x) = -u(x) \), where:

\[
f(x) = b(y - x) + \sum_i c_i(x_t^i)
\]

(18)

\( b: \) convex, continuously differentiable

\( c_i: \) convex, non-differentiable

Consider the following blockwise coordinate descent (BCD) algorithm. Choose an initial guess for \( x \), and set \( i = 1 \). Iterate the following until convergence.

- Optimise \( f(x) \) over \( x_i^i \), holding \( x_i^j \) fixed for all \( j \neq i \). Denote this optimum by \( \hat{x}_i^i \).
- Update \( x \) by setting the coordinates relevant to the \( i \)th asset, \( x_i^i \), equal to \( \hat{x}_i^i \).
- If \( i = N \), set \( i = 1 \); otherwise set \( i = i + 1 \).

Seminal work of Tseng (2001) shows that for \( f(x) \) of the form (18), under fairly mild continuity assumptions, any limit point of the BCD iteration is a minimiser of \( f(x) \). Note that for a generic non-differentiable convex function, there is no reason to expect BCD to find the global minimum and it is trivial to construct examples where it fails to do so for almost any starting point. The key assumption that makes this algorithm work is that the non-differentiable part is separable”, as in (18).

**Intuition 4** The globally optimal multi-asset trading path \( x \) can be found by treating each asset in turn, keeping positions in the others fixed. Each single-asset optimal path is immediately incorporated into \( x \) before proceeding to the next asset.

If \( b(y - x) \) is a quadratic function, such as (16) summed over \( t \), then it projects to a lower-dimensional quadratic when \( x_j^j (j \neq i) \) are held fixed and \( x_i^i \) alone is allowed to vary. In this case, each iteration calls for minimising \( q(x_i^i) + c_i(x_t^i) \) where \( q(x_i^i) \) is quadratic. This subproblem is, mathematically, a single-asset problem, and yet the coefficients of the quadratic function \( q(x_i^i) \) depend on the rest of the portfolio. This is as it should be. Intuitively, increasing holdings of the \( i \)th asset could increase the portfolio risk, or it could actually reduce the portfolio risk if the \( i \)th asset is a hedge. One needs to at least know the risk exposures of the rest of the portfolio when performing optimisation for the \( i \)th asset’s trading path.

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1 This is true for a wide variety of cost functions that have been considered. For example, the model of Kyle & Obizhaeva (2011) has this property, as does borrow cost, market impact as in Almgren et al (2005), and piecewise-linear functions, etc.
Finding optimal paths: one asset, multiple periods

Now let us consider the multiperiod problem for a single asset, in which case the ideal sequence \( y = (y_t) \) and the optimal holdings (or equivalently, hidden states) \( x = (x_t) \) are both univariate time series. Since the multiperiod many asset problem can be reduced to iteratively solving a sequence of single-asset problems, the methods we develop in this section are important even if our main interest is in multi-asset portfolios.

Certain special cases lend themselves to treatment by fast special-purpose optimisers. For example, if all of the terms in (13) happen to be quadratic (i.e., logs of Gaussians) and there are no constraints, then the associated HMM is a linear-Gaussian state space model and the appropriate tool is the Kalman smoother. If the state space is continuous, and if the objective function and all constraints are convex and differentiable, then modern convex solvers apply.

A very important class of examples arises when there are no constraints, but the cost function is a convex and non-differentiable function of the difference \( x_t - x_{t-1} \). This allows for non-quadratic terms, as in Almgren et al. (2005), and non-differentiable terms such as Kyle & Obizhaeva’s (2011) spread term. In this case we can use Tseng’s theorem again, applied to trades rather than positions. Writing \( x_t \) as Kyle & Obizhaeva’s (2011) spread term. In this case we can use the fact that we are working with a state space model of the form (19). The objective function is a convex function of the state variables, and since the state space is continuous, the optimal path can be found using a dynamic programming approach.

\[
\max_{x_t} \sum_{t=1}^{T} c_t(x_t) + \log \sum_{x_T} \pi(x_T) \quad \text{subject to} \quad x_t \leq x_{t-1} \quad \forall t, \quad x_T = \pi(x_T)
\]

where \( c_t(x_t) \) is the cost function at time \( t \), \( \pi(x_T) \) is the initial state probability, and \( x_T \) is the final state of the path.

We then perform coordinate descent over the trades \( \delta_1, \delta_2, \ldots, \delta_T \), using a warm start from a previous optimisation if one is available. Equation (19) satisfies the convergence criteria of Tseng (2001) that the non-differentiable term is separable across time while the non-separable term is differentiable.

We now present a general-purpose method that is slower than the one just described, because it is a Monte Carlo statistical method, but which works for absolutely any cost function (irrespective of differentiability, convexity or other concerns) and any constraints that can be expressed as single-asset constraints. It handles cases where a discrete solution is actually preferred over a continuous one, such as when trading is desired to be in round lots. This method is based on the HMM representation (11), (12) and (13).

Stocks and most other assets trade in integer multiples of a fundamental unit, so the state space is finite, but so large that it is well approximated by a continuous one. Nonetheless, a finite state space could be a useful tool. If the state space were finite, we could follow the Viterbi algorithm to this ‘smaller state space’ gives a good approximation of the optimal path, which becomes a better approximation as more sample paths are added. We chose the model of Kyle & Obizhaeva (2011) for this example because it is theoretically justified by basic microstructure invariance assumptions, and it was fit to a large data set of portfolio transitions, hence it also has empirical support.

For simplicity, we assume the asset being traded is the ‘benchmark asset’, which Kyle & Obizhaeva (2011) use to centre their model. This asset has price \( P = 40 \), daily volatility of \( \sigma = 0.02 \), and average daily volume \( V = 10^6 \) shares. When using the reference asset, the Kyle-Obizhaeva cost function simplifies to:

\[
c_t(\delta_t) = \kappa_1 |\delta_t| + \kappa_2 \delta_t^2 / (0.01PV)
\]

and the optimal path can be found through recursion.
where we take $\kappa_1 = 2.89 \times 10^{-4}$ and $\kappa_2 = 7.91 \times 10^{-4}$. With these parameter settings, the cost to trade 1% of the average daily volume is 10.8 basis points of the trade size. We assume an initial holding of $x_0 = 0$ and take $\gamma = 10^{-5}$ as our risk aversion parameter. Our stylised alpha term structure is a simple sum of two exponential-decay curves: model 1 is initially 25bp, with a half-life of four periods, while model 2 is initially –40bp with a shorter half-life of two periods. Adding these two models produces a term structure that is negative, then positive, then decays to almost zero within about 20 periods.

It is tempting to wonder whether a purely quadratic approximation to cost (ie, $\kappa_1 = 0$ and some other value of $\kappa_2$) might suffice since such a problem could be easily solved by the Kalman smoother. However, such approximations can be misleading. Consider the class of problems with the same $y_T$, $\gamma$ and $\Sigma_T$ as in the numerical example above, but with purely quadratic costs; call this the ‘quadratic class’. We propose that no solution in the quadratic class is particularly close to the solution that globally optimises true utility (ie, utility computed with the non-quadratic cost function) and the closest is the blue line shown in figure 1.

The best possible Kalman path (blue line) places a larger number of trades than the Viterbi path (red line), but the individual trades are smaller. This is because the purely quadratic approximation is over-estimating the cost of large trades and under-estimating the true cost of small trades relative to (21), which happens to be closer to linear in the region of interest. The absolute-value term in (21) allows sparse solutions, as is familiar from elastic-net regression. The particle filter and subsequent Viterbi estimation ran in a few seconds on a notebook computer.

Conclusions
The framework presented here renders multiperiod optimisation and multiperiod tracking problems computationally accessible, even with realistic costs such as bid-ask spread pay and nonlinear market impact. One of the more striking conclusions is that a sequence of single-asset trading path optimisations provably converges to the globally optimal multi-asset solution.

In our view, the main limitations of the model are as follows. We treat the trading cost as non-stochastic, whereas in a real quantitative trading scenario, a meaningful part of the volatility (and higher moments) may arise from randomness in realised trading costs (or ‘slippage’). Liquidity events such as that in August 2007, and the subsequent return of liquidity, provide empirical evidence that realised slippage has meaningful left and right tails. The model as presented here does not take into account investors’ aversion to such higher moments.

In a related limitation, the model is formulated as if we expect to completely fill each order. Hence it does not attempt to find the optimal solution for a passive execution strategy in which order fill percentages are stochastic. This is of great interest, however, as passive strategies typically pay lower costs on the orders that are filled. These represent exciting directions for future research.

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REFERENCES

Almgren R and N Chriss 1999 Value under liquidation Risk December, pages 61–63
Almgren R, C Thum, E Hauptmann and H Li 2005 Equity market impact Risk July, pages 57–62
Bellman R 1957 Dynamic Programming Princeton University Press
Doucet A and AM Johansen 2009 A tutorial on particle filtering and smoothing: fifteen years later Handbook of Nonlinear Filtering, volume 12, pages 656–704 Oxford University Press
Gărleanu N and LH Pedersen 2013 Dynamic trading with predictable returns and transaction costs Journal of Finance 68(6), pages 2,309–2,340
Godsill S, A Doucet and M West 2001 Maximum a posteriori sequence estimation using Monte Carlo particle filters Annals of the Institute of Statistical Mathematics 1(53), pages 82–96
Zakamouline V and S Koekekbakker 2009 A generalisation of the mean-variance analysis European Financial Management 5(15), pages 934–970