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Abstract simplicity of complete Kac–Moody groups over finite fields

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ABSTRACT

Let G be a Kac–Moody group over a finite field corresponding to a generalized Cartan matrix A, as constructed by Tits. It is known that G admits the structure of a BN-pair, and acts on its corresponding building. We study the complete Kac–Moody group $\widehat{G}$ which is defined to be the closure of G in the automorphism group of its building. Our main goal is to determine when complete Kac–Moody groups are abstractly simple, that is have no proper non-trivial normal subgroups. Abstract simplicity of $\widehat{G}$ was previously known to hold when A is of affine type. We extend this result to many indefinite cases, including all hyperbolic generalized Cartan matrices A of rank at least four. Our proof uses Tits' simplicity theorem for groups with a BN-pair and methods from the theory of pro-p groups.

1. Introduction

Let k denote a finite field which will remain fixed throughout the paper. Let A be a generalized Cartan matrix, and let $\mathbb{G}_A$ be the corresponding Kac–Moody group functor of simply-connected type, as constructed by Tits in [25]. The group $G(A) := \mathbb{G}_A(k)$ is usually called a *minimal* or *incomplete* Kac–Moody group.

Distinct constructions of *complete* Kac–Moody groups are given in the papers of Carbone and Garland [4] and Remy and Ronan [20]; in both cases the group constructed is the completion of Tits' group $G(A)$ with respect to a certain topology. In [20], the topology comes from the action of $G(A)$ on its associated positive building; we denote the corresponding completion by $\widehat{G}(A)$.[1] In [4], one starts with the integrable highest weight module V^λ for the Kac–Moody algebra $\mathfrak{g}(A)$ corresponding to a regular dominant integral weight λ, then considers a certain $\mathbb{Z}$-form, $V^\lambda_\mathbb{Z}$ of V^λ, and the action of $G(A)$ on a k-form, V^λ_k of $V^\lambda_\mathbb{Z}$. The corresponding completion of G (which depends on λ) will be denoted by $\widehat{G}^\lambda(A)$. We describe these constructions in detail in Section 2.

Though constructed in different ways, the groups $\widehat{G}(A)$ and $\widehat{G}^\lambda(A)$ have very similar structure. In particular, in both cases the complete Kac–Moody group is locally compact and totally disconnected. Recently, Baumgartner and Remy showed[2] that for any weight λ, the Remy–Ronan completion $\widehat{G}(A)$ is a homomorphic image of the Carbone–Garland completion $\widehat{G}^\lambda(A)$ (see Theorem 2.6 for a precise statement).

The main goal of this paper is to investigate when complete Kac–Moody groups are abstractly simple. Recall that a topological group is called *abstractly simple* if it has no proper non-trivial normal subgroups, and *topologically simple* if it

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[1] Such completions are called *topological Kac–Moody groups* in [20].

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has no proper non-trivial closed normal subgroups. In view of Theorem 2.6, the simplicity question should be asked for the smaller groups $\widehat{G}(A)$. An obvious necessary condition for the simplicity of $\widehat{G}(A)$ is that A should be indecomposable: if $A_1, \ldots, A_k$ are indecomposable blocks of A, then $\widehat{G}(A) \cong \prod_{i=1}^{k} \widehat{G}(A_i)$. Remy [18, Theorem 2.A.1] proved[3] that if A is indecomposable, then the group $\widehat{G}(A)$ is topologically simple when $|k| > 3$, and asked whether abstract simplicity holds as well [19, Question 30]. We answer this question in the affirmative for a large class of Kac–Moody groups.

Theorem 1.1. *Let A be an indecomposable generalized Cartan matrix. Assume that one of the following holds:*

(a) $|k| > 3$, $p = \mathrm{char}(k) > 2$, *A is symmetric, and any 2×2 submatrix of A is of finite or affine type;*
(b) $|k| > 3$ *and any 2×2 submatrix of A is of finite type.*

Then the group $\widehat{G}(A)$ is abstractly simple.

Remark. In the hypotheses of Theorem 1.1, a submatrix is not necessarily proper.

Abstract simplicity of $\widehat{G}(A)$ was previously known only when A is of finite type (in which case $\widehat{G}(A)$ is a finite group) or A is of affine type, in which case $\widehat{G}(A)$ is isomorphic to the group of K-points of a simple algebraic group defined over $K = k((t))$ [18]. The class of matrices covered by Theorem 1.1 includes many indefinite examples, including all hyperbolic matrices of rank at least four — see Proposition 2.1.

Remark. Recently, Caprace and Remy [3] proved (abstract) simplicity of the incomplete group $G(A)$ modulo its finite center in the case when the associated Coxeter group is not affine and assuming that $|k|$ is sufficiently large. If A is affine, the incomplete group $G(A)$ (modulo its center) is infinite and residually finite, and hence cannot be simple.

Briefly, our approach to proving Theorem 1.1 will be as follows. A celebrated theorem of Tits [2] gives sufficient conditions (called "simplicity axioms") for a group with a BN-pair to be simple. The group $\widehat{G}(A)$ routinely satisfies most of these axioms if $|k| > 3$ (for arbitrary A), which already implies topological simplicity of $\widehat{G}(A)$. It is not clear if the remaining axioms hold in general; however, they do hold if the "positive unipotent" subgroup $\widehat{U}(A)$ is topologically finitely generated — see Theorem 4.1 ($\widehat{U}(A)$ is defined as the completion of the group $U(A)$ generated by all positive root subgroups). This follows from the fact that $\widehat{U}(A)$ is a pro-p group and basic properties of pro-p groups (see [7, Chapter 1]).

Thus our main task is to prove (topological) finite generation of $\widehat{U}(A)$ under the hypotheses of Theorem 1.1. If any 2×2 submatrix of A is of finite type and $|k| > 3$, then $U(A)$ is finitely generated (as an abstract group) by a theorem of Abramenko [1], which immediately implies finite generation of $\widehat{U}(A)$. This proves Theorem 1.1(b).

In Section 6, we show (see Theorem 6.1) that $\widehat{U}(A)$ is finitely generated as long as

(a) A is symmetric;
(b) for every 2×2 submatrix C of A, the group $\widehat{U}(C)$ is "well behaved".

The latter is a certain technical condition which is "almost" equivalent to generation by simple root subgroups. In Section 7, we show that the group $\widehat{U}(C)$ is well behaved when C is a 2×2 matrix of finite or affine type (assuming $p > 3$), using an explicit realization of $\widehat{U}(C)$. This completes the proof of Theorem 1.1(a). When C is a 2×2 hyperbolic matrix, we do not know if $\widehat{U}(C)$ is well behaved or even if $\widehat{U}(C)$ is finitely generated. It seems that essentially new ideas are needed to settle this case.

Some conventions about topological groups. Let G be a topological group. By a base (resp. subbase) for the topology of G we will mean a base (resp. subbase) of neighborhoods of the identity. Recall that a topological group G is *topologically generated by a set S* if the subgroup abstractly generated by S is dense in G. When discussing finite generation of pro-p groups, we will always be interested in topological generating sets, so the word 'topological' will often be omitted. Since (infinite) pro-p groups are never finitely generated as abstract groups, this convention should not cause any confusion.

2. Kac–Moody algebras and groups

2.1. Generalized Cartan matrices

Let $I = \{1, 2, \ldots, l\}$ be a finite set. A matrix $A = (a_{ij})_{i,j \in I}$ is called a *generalized Cartan matrix* if its entries satisfy the following conditions:

(a) $a_{ij} \in \mathbb{Z}$ for $i, j \in I$, (b) $a_{ii} = 2$, for $i \in I$
(c) $a_{ij} \leq 0$ if $i \neq j$ (d) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

By a *submatrix* of A, we mean a matrix of the form

$$A_J = (a_{ij})_{i,j \in J},$$

[3] Remy considers a more general class of quasi-split Kac–Moody groups.

where J is a subset of I. We say that the submatrix A_J is *proper* if $J \neq I$. The matrix A is called *indecomposable* if there is no partition of the set I into two non-empty subsets so that $a_{ij} = 0$ whenever i belongs to the first subset, while j belongs to the second. A submatrix A_J is called an *indecomposable block* of A if A_J is indecomposable, and J is maximal with this property.

A generalized Cartan matrix A is said to be of

finite (classical) type if A is positive definite; in this case, A is the Cartan matrix of a finite-dimensional semi-simple Lie algebra,
affine type if A is positive semi-definite, but not positive definite,
indefinite type if A is neither of finite nor affine type.

An indefinite matrix is said to be of *hyperbolic type* (in the sense of [9, 5.10, p. 66]) if every proper indecomposable submatrix of A is of finite or affine type.

Proposition 2.1. *Let A be an indecomposable $l \times l$ matrix, and let B be an $s \times s$ submatrix of A. Assume that either*

(i) *A is of finite type,*
(ii) *A is of affine type and $s \leq l - 1$ (that is, B is a proper submatrix of A),*
(iii) *A is of hyperbolic type and $s \leq l - 2$.*

Then B is of finite type.

Proof. Case (i) is obvious, and case (ii) is well known [9, Chapter 4]. Now let A be hyperbolic and $s \leq l - 2$. We can assume that B is indecomposable, since if every indecomposable block of B is of finite type, then so is B. Since A is indecomposable, there exists an indecomposable $(s + 1) \times (s + 1)$ submatrix C of A such that C contains B. Since $s + 1 < l$, the matrix C must be of finite or affine type, whence B is of finite type by cases (i) and (ii). □

2.2. Kac–Moody algebras

For the rest of this section, fix a generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$, and let $l = |I|$. A *realization* of A over $\mathbb{Q}$ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where $\mathfrak{h}$ is a vector space over $\mathbb{Q}$ of dimension $2l - \operatorname{rank}(A)$, and $\Pi = \{\alpha_1, \ldots, \alpha_l\} \subseteq \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_l^\vee\} \subseteq \mathfrak{h}$ are linearly independent sets, such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for $i, j \in I$. As usual, $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. Elements of Π are called *simple roots* and elements of $\Pi^\vee$ *simple coroots*.

The associated Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}_A$ is a Lie algebra over $\mathbb{Q}$, generated by $\mathfrak{h}$ and elements $(e_i)_{i\in I}$, $(f_i)_{i\in I}$ subject to the Serre–Kac relations:

(1) $[\mathfrak{h}, \mathfrak{h}] = 0$ (5) $[e_i, f_j] = 0, \quad i \neq j$
(2) $[h, e_i] = \langle \alpha_i, h \rangle e_i, \quad h \in \mathfrak{h}$ (6) $(\operatorname{ad} e_i)^{-a_{ij}+1}(e_j) = 0, \quad i \neq j$
(3) $[h, f_i] = -\langle \alpha_i, h \rangle f_i, \quad h \in \mathfrak{h}$ (7) $(\operatorname{ad} f_i)^{-a_{ij}+1}(f_j) = 0, \quad i \neq j$
(4) $[e_i, f_i] = \alpha_i^\vee$.

It is easy to see that $\mathfrak{g}_A$ depends only on A and not on its realization (see [9, 1.1]).

Relative to $\mathfrak{h}$, the Lie algebra $\mathfrak{g}$ has decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigsqcup_{\alpha\in\Delta} \mathfrak{g}^\alpha$, where

$$\mathfrak{g}^\alpha \quad = \quad \{x \in \mathfrak{g} \mid [h, x] = \langle \alpha, h \rangle x, \ h \in \mathfrak{h}\},$$

and $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}^\alpha \neq 0\}$. Elements of Δ are called the *roots* of $\mathfrak{g}$. Each root has the form $\sum_{i\in I} n_i \alpha_i$ where $n_i \in \mathbb{Z}$ and either $n_i \geq 0$ for all i, or $n_i \leq 0$ for all i. The roots are called positive or negative accordingly; the set of positive (resp. negative) roots will be denoted by Δ^+ (resp. Δ^-). The *height* of a root $\alpha = \sum_{i=1}^l n_i \alpha_i$ is defined to be the integer $\sum_i n_i$.

2.3. Real roots and the Weyl group

For $i \in I$ define $w_i \in \operatorname{Aut}(\mathfrak{h}^*)$ by setting $w_i(\alpha) = \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i$. The group $W = \langle \{w_i\} \rangle$ generated by the w_i is called the *Weyl group* associated to A. The set $\Phi = W(\Pi)$ is a subset of Δ, called the set of *real roots*. The remaining roots $\Delta \setminus \Phi$ are called *imaginary roots*.

The Weyl group W has a faithful action on $\mathfrak{h}$ defined by $w_i(h) = h - \langle \alpha_i, h \rangle \alpha_i^\vee$. Moreover, the pairing $\langle \cdot, \cdot \rangle$ is W-invariant, that is, $\langle w\alpha, wh \rangle = \langle \alpha, h \rangle$ for $\alpha \in \mathfrak{h}^*$, $h \in \mathfrak{h}$ and $w \in W$.

For each real root α, define the corresponding coroot $\alpha^\vee$ as follows: write α in the form $w\alpha_i$ for some $w \in W$ and $i \in I$ and set $\alpha^\vee = w\alpha_i^\vee$. One can show (see [9]) that $\alpha^\vee$ is independent of the above choice. The correspondence $\alpha \mapsto \alpha^\vee$ is not linear; however, it does satisfy some nice properties:

Proposition 2.2. *The following hold:*

(a) *For each $\alpha \in \Phi$, the coroot $\alpha^\vee$ is an integral linear combination of $\{\alpha_i^\vee\}$, and the coefficients are all non-negative (resp. non-positive) if $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$). Furthermore, $(-\alpha)^\vee = -\alpha^\vee$.*
(b) *Given $\alpha, \beta \in \Phi$, we have $\langle \alpha, \beta^\vee \rangle > 0$ (resp. $= 0, < 0$) $\Leftrightarrow$ $\langle \beta, \alpha^\vee \rangle > 0$ (resp. $= 0, < 0$).*
(c) *For every $\alpha \in \Phi$, we have $\langle \alpha, \alpha^\vee \rangle = 2$.*

In order to define Kac–Moody groups, we introduce a related group $W^* \subseteq \mathrm{Aut}(\mathfrak{g})$. By definition, W^* is generated by elements $\{w_i^*\}_{i\in I}$, where

$$w_i^* = \exp(\mathrm{ad}\, e_i)\exp(-\mathrm{ad}\, f_i)\exp(\mathrm{ad}\, e_i) = \exp(-\mathrm{ad}\, f_i)\exp(\mathrm{ad}\, e_i)\exp(-\mathrm{ad}\, f_i).$$

The group W^* is a central extension of W. More specifically, there is a surjective homomorphism $\varepsilon : W^* \to W$ which sends w_i^* to w_i for all i; the kernel of ε is an elementary abelian group of exponent 2 generated by $\{(w_i^*)^2\}$, as follows immediately from [25, 3.3].

Finally, we define certain elements $\{e_\alpha \in \mathfrak{g}\}_{\alpha\in\Phi}$. Given $\alpha \in \Phi$, write α in the form $w\alpha_j$ for some $j \in I$ and $w \in W$, choose $w^* \in W^*$ which maps onto w, and set $e_\alpha = w^* e_{\alpha_j}$. It is clear from [25, (3.3.2)] that e_α belongs to g^α, e_α is uniquely determined up to sign, and for all $i \in I$, $w_i^* e_\alpha = \eta_{\alpha,i} e_{w_i\alpha}$ for some constants $\eta_{\alpha,i} \in \{\pm 1\}$. These constants $\{\eta_{\alpha,i}\}$ will appear in the definition of Kac–Moody groups.

2.4. Kac–Moody groups and Tits' presentation

The construction of (incomplete) Kac–Moody groups over arbitrary fields is due to Tits [25]. One may define these groups by generators and relators. While not explicitly stated in Tits' paper, such a presentation appears in the papers of Carter [5] and (in a slightly different form) Morita and Rehmann [13].

The group $G = G(A)$ defined below is called the *incomplete simply-connected Kac–Moody group* corresponding to A. The presentation we use is "almost canonical" except for the choice of elements $\{e_\alpha\}$ which determine the constants $\{\eta_{\alpha,i}\}$.

By definition, $G(A)$ is generated by the set of symbols $\{\chi_\alpha(u) \mid \alpha \in \Phi, u \in k\}$ satisfying relations (R1)–(R7). In all the relations i, j are elements of I, u, v are elements of k (arbitrary, unless mentioned otherwise) and α and β are real roots.

(R1) $\chi_\alpha(u+v) = \chi_\alpha(u)\chi_\alpha(v)$;

(R2) Let (α, β) be a *prenilpotent pair,* that is, there exist $w,\ w' \in W$ such that

$$w\alpha,\ w\beta \in \Phi^+ \quad \text{and} \quad w'\alpha,\ w'\beta \in \Phi^-.$$

Then

$$[\chi_\alpha(u), \chi_\beta(v)] = \prod_{m,n\geq 1} \chi_{m\alpha+n\beta}(C_{mn\alpha\beta}u^m v^n),$$

where the product on the right-hand side is taken over all real roots of the form $m\alpha + n\beta$, $m, n \geq 1$, in some fixed order, and $C_{mn\alpha\beta}$ are integers independent of k (but depending on the order).

For each $i \in I$ and $u \in k^*$ set

$$\chi_{\pm i}(u) = \chi_{\pm\alpha_i}(u),$$
$$\widetilde{w}_i(u) = \chi_i(u)\chi_{-i}(-u^{-1})\chi_i(u),$$
$$\widetilde{w}_i = \widetilde{w}_i(1) \quad \text{and} \quad h_i(u) = \widetilde{w}_i(u)\widetilde{w}_i^{-1}.$$

The remaining relations are

(R3) $\widetilde{w}_i\chi_\alpha(v)\widetilde{w}_i^{-1} = \chi_{w_i\alpha}(\eta_{\alpha,i}u)$,
(R4) $h_i(u)\chi_\alpha(v)h_i(u)^{-1} = \chi_\alpha(vu^{-\langle\alpha,\alpha_i^\vee\rangle})$ for $u \in k^*$,
(R5) $\widetilde{w}_i h_j(u)\widetilde{w}_i^{-1} = h_j(u)h_i(u^{-a_{ji}})$,
(R6) $h_i(uv) = h_i(u)h_i(v)$ for $u, v \in k^*$, and
(R7) $[h_i(u), h_j(v)] = 1$ for $u, v \in k^*$.

An immediate consequence of relations (R3) is that $G(A)$ is generated by $\{\chi_{\pm i}(u)\}$.

Remark. In [10, Proposition 2.3], it is shown that a pair $\{\alpha, \beta\}$ is prenilpotent if and only if $\alpha \neq -\beta$ and $|(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Phi| < \infty$. Thus the product on the right-hand side of (R2) is finite.

Intuitively, one should think of the above presentation as an analogue of the Steinberg presentation for classical groups with $\chi_\alpha(u)$ playing the role of $\exp(ue_\alpha)$. In the next subsection we give a representation-theoretic interpretation of Kac–Moody groups which makes the above analogy precise.

Next we introduce several subgroups of $G = G(A)$:

1. *Root subgroups* U_α. For each $\alpha \in \Phi$ let $U_\alpha = \{\chi_\alpha(u) \mid u \in k\}$. By relations (R1), each U_α is isomorphic to the additive group of k.
2. *The "extended" Weyl group* $\widetilde{W}$. Let $\widetilde{W}$ be the subgroup of G generated by elements $\{\widetilde{w}_i\}_{i\in I}$. One can show that $\widetilde{W}$ is isomorphic to the group W^* introduced before, so there is a surjective homomorphism $\varepsilon : \widetilde{W} \to W$ such that $\varepsilon(\widetilde{w}_i) = w_i$ for $i \in I$. Given $\widetilde{w} \in \widetilde{W}$ and $w \in W$, we will say that $\widetilde{w}$ is a representative of w if $\varepsilon(\widetilde{w}) = w$. It will be convenient to identify (non-canonically) W with a subset (not a subgroup) of $\widetilde{W}$ which contains exactly one representative of every element of W. By abuse of notation, the set of those representatives will also be denoted by W. It follows from relations (R3) that $wU_\alpha w^{-1} = U_{w\alpha}$ for any $\alpha \in \Phi$ and $w \in W$.

3. *"Unipotent" subgroups.* Let $U^+ = \langle U_\alpha \mid \alpha \in \Phi^+ \rangle$, and $U^- = \langle U_\alpha \mid \alpha \in \Phi^- \rangle$.
4. *"Torus" ("diagonal" subgroup).* Let $H = \langle \{h_i(u) \mid i \in I, u \in k\} \rangle$. One can show that relations (R6)–(R7) are defining relations for H, so H is isomorphic to the direct sum of l copies of k^*.
5. *"Borel" subgroups.* Let $B^+ = \langle U^+, H \rangle$ and $B^- = \langle U^-, H \rangle$. By relations (R4), H normalizes both U^+ and U^-, so we have $B^+ = HU^+ = U^+H$ and $B^- = HU^- = U^-H$.
6. *"Normalizer."* Let N be the subgroup generated by $\widetilde{W}$ and H. Since $\widetilde{W}$ normalizes H, we have $N = \widetilde{W}H$. It is also easy to see that $N/H \cong W$.

Tits [25] proved that (B^+, N) and (B^-, N) are BN-pairs[4] of G. In fact, G admits the stronger structure of a twin BN-pair, but we will not use it. Let X^+ and X^- be the buildings associated with (B^+, N) and (B^-, N), respectively. Since the field k is finite, the buildings X^+ and X^- are locally finite as chamber complexes. In fact, X^+ and X^- have constant thickness $|k| + 1$ (see [6, Appendix KMT]).

Below we list some of the fundamental properties of these BN-pairs. We will work mostly with the positive BN-pair (B^+, N), and from now on, write B for B^+ and U for U^+.

(a) $B \cap N = H$, so the Coxeter group associated to (B, N) is isomorphic to the Weyl group $W = W(A)$;
(b) Bruhat decomposition: $G = BWB$;
(c) Birkhoff decomposition: $G = U^-WB = B^-WU = UWB^- = BWU^-$.

Of course, (b) follows directly from (B, N) being a BN-pair, and the proof of (c) uses the twin BN-pair structure (see [10]). Finally, we shall need a presentation by generators and relators for the group U established by Tits [24, Proposition 5].

Theorem 2.3. *The group U is generated by the elements $\{\chi_\alpha(u) \mid \alpha \in \Phi^+, u \in k\}$ subject to relations* (R1) *and* (R2) *defined earlier in this section.*

2.5. Representation-theoretic interpretation of Kac–Moody groups

The following interpretation of Kac–Moody groups was given by Carbone and Garland [4] (see also [26]). This construction generalizes that of Chevalley groups [22]. Let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{g}$. Let $\Lambda \subseteq \mathfrak{h}^*$ be the linear span of α_i, for $i \in I$, and $\Lambda^\vee \subseteq \mathfrak{h}$ be the linear span of $\alpha_i^\vee$, for $i \in I$. Let $\mathcal{U}_\mathbb{Z} \subseteq \mathcal{U}$ be the $\mathbb{Z}$-subalgebra generated by $e_i^m/m!$, $f_i^m/m!$, and $\binom{h}{m}$, for $i \in I$, $h \in \Lambda^\vee$ and $m \geq 0$. Then $\mathcal{U}_\mathbb{Z}$ is a $\mathbb{Z}$-form of $\mathcal{U}$, i.e. $\mathcal{U}_\mathbb{Z}$ is a subring and the canonical map $\mathcal{U}_\mathbb{Z} \otimes \mathbb{Q} \longrightarrow \mathcal{U}$ is bijective. For a field K, let $\mathcal{U}_K = \mathcal{U}_\mathbb{Z} \otimes K$, and $\mathfrak{g}_K = \mathfrak{g}_\mathbb{Z} \otimes K$.

Now let $\lambda \in \mathfrak{h}^*$ be a regular dominant integral weight, that is, $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{>0}$ for every $i \in I$. Let V^λ be the corresponding irreducible highest weight module. Choose a highest weight vector $v_\lambda \in V^\lambda$, and let $V_\mathbb{Z}^\lambda \subset V^\lambda$ be the orbit of v_λ under the action of $\mathcal{U}_\mathbb{Z}$. Then $V_\mathbb{Z}^\lambda$ is a $\mathbb{Z}$-form of V_λ as well as a $\mathcal{U}_\mathbb{Z}$-module. Similarly, $V_k^\lambda := k \otimes_\mathbb{Z} V_\mathbb{Z}^\lambda$ is a $\mathcal{U}_k$-module.

It is straightforward to establish the following (see [26, Proposition 3]).

Proposition 2.4. *There is a (unique) homomorphism $\pi_\lambda : G \to \mathrm{Aut}(V_k^\lambda)$ such that*

$$\pi_\lambda(\chi_{\alpha_i}(u)) = \sum_{m=0}^{\infty} u^m \frac{e_i^m}{m!} \quad \text{for } i \in I \text{ and } u \in k,$$

$$\pi_\lambda(\chi_{-\alpha_i}(u)) = \sum_{m=0}^{\infty} u^m \frac{f_i^m}{m!} \quad \text{for } i \in I \text{ and } u \in k.$$

The expressions on the right-hand side are well-defined automorphisms of V_k^λ since e_i and f_i are locally nilpotent on V_k^λ. Let $G^\lambda = \pi_\lambda(G)$. As we will see later in this section, the kernel of π_λ is finite, central and contained in H.

2.6. Complete Kac–Moody groups

As mentioned in the introduction, distinct completions of $G(A)$ were given in the papers of Carbone and Garland [4] and Remy and Ronan [20]. We now briefly review these constructions, starting with the Remy–Ronan completion. As above, let X^+ be the building associated with the positive BN-pair (B, N), and consider the action of G on X^+. Recall that X^+ is locally finite as a chamber complex. Define the topology on G by the subbase[5] consisting of stabilizers of vertices of X^+ or, equivalently, fixators (pointwise stabilizers) of chambers of X^+. We shall call this topology the *building topology*. The completion of G in its building topology will be referred to as the *Remy–Ronan completion* and denoted by $\widehat{G}$.

We will make few references to the action of G on its building in this work. All we will need is the description of the building topology in purely group-theoretic terms. Since (B, N) is a BN-pair, we know that

(a) The subgroup B of G is a chamber fixator,
(b) G acts transitively on the set of chambers of X^+.

Therefore, the family $\{gBg^{-1}\}_{g \in G}$ is a subbase for the building topology.

[4] Recall that BN-pairs are also called Tits systems.

[5] Recall that by 'subbase' for a topology on a group, we mean a 'subbase of neighborhoods of the identity'.

Let Z be the kernel of the natural map $G \to \widehat{G}$ (or, equivalently, the kernel of the action of G on X^+). Using results of Kac and Peterson [10], Remy and Ronan [20, 1.B] showed that Z is a subgroup of H (and hence finite); furthermore, Z coincides with the center of G.

Now let $\widehat{B}$ (resp. $\widehat{U}$) be the closure of B (resp. U) in $\widehat{G}$. The natural images of N and H in $\widehat{G}$ are discrete, and therefore we will denote them by the same symbols (without hats). This involves some abuse of notation since the image of H in $\widehat{G}$ is isomorphic to H/Z.

The following theorem is a collection of results from [17,20]:

Theorem 2.5. *Let $\widehat{G}$, $\widehat{B}$ and N be as above. The following hold:*

(a) *The pair $(\widehat{B}, N)$ is a BN-pair of $\widehat{G}$. Moreover, if $\widehat{X}^+$ is the associated building, there exists a $\widehat{G}$-equivariant isomorphism between X^+ and $\widehat{X}^+$. In particular, the Coxeter group associated to $(\widehat{B}, N)$ is isomorphic to $W = W(A)$.*

(b) *The group $\widehat{B}$ is an open profinite subgroup of $\widehat{G}$. Furthermore, $\widehat{U}$ is an open pro-p subgroup of $\widehat{B}$.*

Now we turn to the Carbone–Garland completion. Let λ be a regular weight, and let $G^\lambda = G^\lambda(A)$ and V_k^λ be defined as in the previous subsection. Now we define the *weight topology* on G^λ by taking stabilizers of elements of V_k^λ as a subbase of neighborhoods of the identity. The completion of G^λ in this topology will be referred to as the *Carbone–Garland completion* and denoted by $\widehat{G}^\lambda(A)$. Since $G^\lambda(A)$ is a homomorphic image of $G(A)$, we can think of $\widehat{G}^\lambda(A)$ as a completion of G (and not G^λ). Let $\widehat{B}^\lambda$ (resp. $\widehat{U}^\lambda$) be the closures of B (resp. U) in $\widehat{G}^\lambda(A)$. Then the obvious analogue of Theorem 2.5 holds; the fact that $\widehat{U}^\lambda$ is a pro-p group will be proved at the end of this section (see Proposition 2.7); for all other assertions see [4, Section 6].

The following relationship between the Remy–Ronan and Carbone–Garland completions was established[6] by Baumgartner and Remy:

Theorem 2.6. *For any regular weight λ, there exists a (canonical) continuous surjective homomorphism $\varepsilon_\lambda : \widehat{G}^\lambda \to \widehat{G}$. The kernel K_λ of ε_λ is equal to $\bigcap_{g\in\widehat{G}} g\widehat{B}^\lambda g^{-1}$.*

It follows from Theorem 2.6 that the kernel of the map $\pi_\lambda : G \to G^\lambda$ is finite and central. Indeed, consider the sequence of homomorphisms

$$G \xrightarrow{\pi_\lambda} G^\lambda \longrightarrow \widehat{G}^\lambda \xrightarrow{\varepsilon_\lambda} \widehat{G}$$

Clearly, the composition of these three maps is the natural map from G to $\widehat{G}$. We know that the kernel of the latter map is finite and central, hence the same should be true for π_λ.

In the case when A is an affine matrix, Garland [8] showed that K_λ is a central subgroup of $H \subseteq \widehat{B}^\lambda$ (and hence finite). It is not clear to us how large K_λ can be in general. Since $\widehat{B}^\lambda$ is a profinite group, so is K_λ; furthermore, K_λ has a finite index pro-p subgroup, which follows from Proposition 2.7.

Proposition 2.7. *Let $\widehat{U}^\lambda$ be the closure of U in $\widehat{G}^\lambda$. The group $\widehat{U}^\lambda$ is a pro-p group.*

Proof. In [4], it is shown that the k-vector space V_k^λ admits a basis $\Psi = \{v_1, v_2, v_3, \ldots\}$ consisting of weight vectors, that is, for each $i \in \mathbb{N}$ there exists a weight μ_i of V^λ such that v_i lies in the weight component $V^\lambda_{\mu_i}$. Each weight μ of V^λ is of the form $\mu = \lambda - \sum_{i=1}^{l} k_i\alpha_i$, where $k_i \in \mathbb{Z}_{\geq 0}$. Define the *depth* of μ to be $\mathrm{depth}(\mu) = \sum_{i=1}^{l} k_i$. For convenience, we order the elements of Ψ such that $\mathrm{depth}(\mu_i) \leq \mathrm{depth}(\mu_j)$ if $i < j$.

For each $n \geq 1$ let V_n be the k-span of the set $\{v_1, v_2, \ldots, v_n\}$. The group U stabilizes V_n; moreover, it acts by upper-unitriangular matrices (with respect to the above basis). Therefore, we have a homomorphism $\pi_n : U \to GL_n(k)$ whose image is a finite p-group (since k has characteristic p). Then $U_n = \mathrm{Ker}\,\pi_n$ consists of elements of U which fix V_n pointwise. Since $\bigcup_{n\geq 1} V_n = V_k^\lambda$, the groups $\{U_n\}_{n=1}^\infty$ form a base for the weight topology on U. Since each U_n is a normal subgroup of U of p-power index, the completion of U with respect to the weight topology is a pro-p group. □

Remark. Remy and Ronan [20] prove that $\widehat{U}$ is a pro-p group using its action on the building of $\widehat{G}$. This fact can also be deduced from Proposition 2.7 by applying Theorem 2.6.

We finish this section by describing explicitly the groups $G(A)$, $\widehat{G}(A)$ and $\widehat{G}^\lambda(A)$ in the special case $A = A^{(1)}_{d-1}$ for some $d \geq 2$ (in the notation of [9, Chapter 4]); an analogous result holds for any affine matrix A — see [8]. In this case, the incomplete group $G(A)$ is isomorphic to a central extension of the group $SL_d(k[t, t^{-1}])$ by k^*. The Remy–Ronan completion $\widehat{G}(A)$ is isomorphic to $PSL_d(k((t)))$. It is easy to see that the building topology on $\widehat{G}(A)$ coincides with the topology on $PSL_d(k((t)))$ induced from the local field $k((t))$.

[6] Private communication.

Recall (see above) that the center of $G(A)$ always lies in the kernel of the natural map $G(A) \to \widehat{G}(A)$. On the other hand, the center of $G(A)$ does usually have non-trivial image in the Carbone–Garland completion $\widehat{G}^{\lambda}(A)$. For any λ, there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k^* & \longrightarrow & \widehat{SL_d}\,(k((t))) & \longrightarrow & SL_d\,(k((t))) & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle \rho_\lambda} & & \downarrow{\scriptstyle \rho} & & \\ & & & & \widehat{G}^{\lambda}(A) & \xrightarrow{\varepsilon_\lambda} & \widehat{G}(A) & & \end{array}$$

where $\widehat{SL_d}(k((t)))$ is the universal central extension of $SL_d(k((t)))$, the top row of the diagram is exact, and the homomorphisms ρ, ρ_λ and ε_λ are surjective (ρ is composition of the natural map from $SL_d(k((t)))$ to $PSL_d(k((t)))$ and an isomorphism between $PSL_d(k((t)))$ and $\widehat{G}(A)$). The map ρ_λ may or may not be an isomorphism depending on λ (see [8, Chapter 12]).

Notational remark. If A is a generalized Cartan matrix, the notations $G(A)$, $\widehat{G}(A)$, $U(A)$ etc. introduced in this section will have the same meaning throughout the paper. The reference to A will be omitted when clear from the context. The last remark does not apply to Section 3 where G stands for an arbitrary group.

3. Tits' abstract simplicity theorem

The following is a statement of the Tits simplicity theorem for groups with a BN-pair (see [2, Ch. IV, No. 2.7]).

Theorem 3.1. *Consider a quadruple (G, B, N, U) where G is a group, (B, N) is a BN-pair of G whose associated Coxeter system is irreducible, and $U \leq B$ is a subgroup whose G-conjugates generate the entire group G.*

Assume the following:

(a) *U is normal in B and $B = UH$, where $H = B \cap N$.*
(b) *$[G, G] = G$.*
(c) *If Λ is a proper normal subgroup of U, then $[U/\Lambda,\ U/\Lambda] \neq U/\Lambda$.*

Let $Z = \bigcap_{g\in G} gBg^{-1}$. Then the group G/Z is abstractly simple.

We remark that Tits' theorem has the following "topological" version whose proof is identical to the "abstract" version.

Theorem 3.2. *Let G, B, N, Z be as above. Assume that G is a topological group, and B is a closed subgroup of G. Let U be a closed subgroup of B, and assume that G is topologically generated by the conjugates of U in G. Assume condition* (a) *above and replace* (b) *and* (c) *by conditions* (b′) *and* (c′)*:*

(b′) *$[G, G]$ is dense in G.*
(c′) *If Λ is a proper normal* **closed** *subgroup of U, then $[U/\Lambda,\ U/\Lambda] \neq U/\Lambda$.*

Then G/Z is topologically simple.

4. Simplicity of complete Kac–Moody groups via Tits' theorem

In this section G will denote an incomplete Kac–Moody group constructed from an **indecomposable** generalized Cartan matrix A. Let B, N, U be as in Section 2, let $\widehat{G}$ be the Remy–Ronan completion of G, and let $\widehat{B}$ (resp. $\widehat{U}$) be the closures of B and (resp. U) in $\widehat{G}$. We shall analyze the conclusion of Tits' theorem applied to the quadruple $(\widehat{G}, \widehat{B}, N, \widehat{U})$.

The group $Z := \bigcap_{g\in\widehat{G}} g\widehat{B}g^{-1}$ is easily seen to be trivial (see Lemma 4.4(e)) and therefore, $\widehat{G}$ is abstractly (resp. topologically) simple provided the hypotheses of Theorem 3.1 (resp. Theorem 3.2) are satisfied.

We will show that the hypotheses of Theorem 3.2 are satisfied provided $|k| > 3$, thus giving a slightly different proof of Remy's theorem on topological simplicity of $\widehat{G}$ [18, Theorem 2.A.1]. We will also prove that hypotheses (b) and (c) of Theorem 3.1 are satisfied as long as $\widehat{U}$ is (topologically) finitely generated. These results will follow from Lemmas 4.4 and 4.5. Thus we will obtain the desired sufficient condition for abstract simplicity of $\widehat{G}$:

Theorem 4.1. *Assume that $|k| > 3$. If $\widehat{U}$ is topologically finitely generated, then $\widehat{G}$ is abstractly simple.*

As an immediate consequence of Theorem 4.1, we deduce part (b) of Theorem 1.1:

Corollary 4.2. *Assume that $|k| > 3$ and any 2×2 submatrix of A is of finite type. Then the complete Kac–Moody group $\widehat{G}(A)$ is abstractly simple.*

Proof. By a theorem of Abramenko [1], under the above assumptions on k and A, the incomplete group $U = U(A)$ is finitely generated (as an abstract group). Thus, $\widehat{U}$ is automatically topologically finitely generated, and therefore $\widehat{G}$ is abstractly simple by Theorem 4.1. □

Before verifying Tits' simplicity axioms for the quadruple $(\widehat{G}, \widehat{B}, N, \widehat{U})$, we obtain an auxiliary result about incomplete groups.

Lemma 4.3. *The following hold:*

(a) *The group G is generated by conjugates of U.*
(b) *If $|k| > 3$, then $[G, G] = G$.*

Proof. (a) We know that G is generated by root subgroups $\{U_{\pm\alpha_i}\}_{i\in I}$ (recall that $\{\alpha_i\}$ are simple roots). Since $U_{\alpha_i} \subset U$ and $w_i U_{\alpha_i} w_i^{-1} = U_{w_i\alpha_i} = U_{-\alpha_i}$, conjugates of U generate G.

(b) Let $i \in I$ and $u \in k$, and let $g = \chi_i(u)$. Choose $t \in k^*$ such that $t^2 \neq 1$ (this is possible since $|k| > 3$), and let $v = u/(t^2-1)$. We have

$$\chi_i(u) = \chi_i\left((t^{\langle\alpha_i,\alpha_i^\vee\rangle} - 1)\, v\right) = \chi_i\left(t^{\langle\alpha_i,\alpha_i^\vee\rangle} v\right) \chi_i(-v) = h_i(t)\chi_i(v)h_i(t)^{-1}\chi_i(v)^{-1} = [h_i(t), \chi_i(v)].$$

So, $\chi_i(u) \in [G, G]$, and similarly one shows that $\chi_{-i}(u) \in [G, G]$. Therefore, $[G, G]$ contains a generating set for G. □

Now we are ready to establish Theorem 4.1 and the corresponding statement about topological simplicity of $\widehat{G}$. By Theorems 3.1 and 3.2, it suffices to prove the following two results[7]:

Lemma 4.4. *The following hold:*

(a) *$\widehat{G}$ is generated by conjugates of $\widehat{U}$.*
(b) *$[\widehat{G}, \widehat{G}]$ is dense in $\widehat{G}$. Moreover, $[\widehat{G}, \widehat{G}] = \widehat{G}$ if $\widehat{U}$ is finitely generated.*
(c) *$\widehat{U}$ is normal in $\widehat{B}$ and $\widehat{B} = \widehat{U}(\widehat{B} \cap N)$.*
(d) *The Coxeter system of the BN-pair $(\widehat{B}, N)$ is irreducible.*
(e) *The group Z is trivial.*

Lemma 4.5. *Let Λ be a proper normal subgroup of $\widehat{U}$. If*

(a) *Λ is closed, or*
(b) *$\widehat{U}$ is finitely generated,*

then

$$[\widehat{U}/\Lambda, \quad \widehat{U}/\Lambda] \neq \widehat{U}/\Lambda.$$

The proofs of Lemmas 4.4 and 4.5 are based on the following properties of pro-p groups.

Proposition 4.6. *Let K be a pro-p group, and let K^* be the closure of $[K, K]K^p$ in K (where K^p is the subgroup generated by pth-powers of elements of K).*

(a) *A subset X of K generates K (topologically) if and only if the image of X in K/K^* generates K/K^*.*
(b) *Suppose that K is finitely generated. Then any two minimal generating sets of K have the same cardinality.*

Proof. (a) is proved in [21, Proposition 25]; it also follows from [7, Proposition 1.9(iii)] and [7, Proposition 1.13]. To prove (b) note that if K is finitely generated, then K/K^* can be viewed as a finite-dimensional space over $\mathbb{F}_p$; let d be the dimension of this space. Let X be a generating set of K. Clearly, $|X| \geq d$. On the other hand, there exists a subset Y of X, with $|Y| = d$ such that the image of Y in K/K^* is a basis of K/K^*. By (a), Y generates K. Thus any minimal generating set of K has cardinality d. □

Proposition 4.7. *Let K be a pro-p group generated by a finite set $\{a_1, \ldots, a_d\}$. Then any element of $[K, K]$ can be written in the form $[a_1, g_1][a_2, g_2] \ldots [a_d, g_d]$ for some $g_1, \ldots, g_d \in K$. In particular, $[K, K]$ is closed.*

Proof. The second assertion of Proposition 4.7 is the statement of [7, Proposition 1.19]. The first assertion is established in the course of the proof of [7, Proposition 1.19]. □

Now we are ready to establish Lemmas 4.4 and 4.5. It will be convenient to prove Lemma 4.5 first.

Proof of Lemma 4.5. Suppose that $[\widehat{U}/\Lambda, \widehat{U}/\Lambda] = \widehat{U}/\Lambda$ or, equivalently, that $\widehat{U} = \Lambda[\widehat{U}, \widehat{U}]$. Then Λ generates $\widehat{U}$ modulo $[\widehat{U}, \widehat{U}]$. Since $\widehat{U}$ is a pro-p group, Λ generates $\widehat{U}$ (topologically) by Proposition 4.6(a). If Λ is closed, then $\Lambda = \widehat{U}$, so Λ is not proper. Thus we proved the desired result, assuming (a).

Now assume (b) that $\widehat{U}$ is finitely generated. Let $\{a_1, a_2, \ldots, a_d\}$ be a finite generating set of $\widehat{U}$ contained in Λ (such a set exists by Proposition 4.6(b)). Applying Proposition 4.7 with $K = \widehat{U}$, we see that $\Lambda \supseteq [\widehat{U}, \widehat{U}]$ (since Λ is normal in $\widehat{U}$). But $\widehat{U} = \Lambda[\widehat{U}, \widehat{U}]$, so we conclude that $\widehat{U} = \Lambda$, a contradiction. □

[7] The first assertion of Lemmas 4.4(b) and 4.5 are not needed for the proof of Theorem 4.1.

Proof of Lemma 4.4. (a) Let G_1 be the subgroup of $\widehat{G}$ generated by conjugates of $\widehat{U}$. By Lemma 4.3, G_1 contains G, whence G_1 is dense in $\widehat{G}$. But G_1 is also open (hence closed) in $\widehat{G}$ since $G_1 \supseteq \widehat{U}$. Therefore, G_1 must be equal to $\widehat{G}$.

(b) The density of $[\widehat{G}, \widehat{G}]$ is clear since $[\widehat{G}, \widehat{G}] \supseteq [G, G] = G$ and G is dense in $\widehat{G}$. Now assuming that $\widehat{U}$ is finitely generated, we shall prove that $[\widehat{G}, \widehat{G}]$ is also open in $\widehat{G}$. Since $\widehat{U}$ is a finitely generated pro-p group, its commutator subgroup $[\widehat{U}, \widehat{U}]$ is closed by Proposition 4.7. So, $\widehat{U}/[\widehat{U}, \widehat{U}]$ is also a finitely generated (abelian) pro-p group. On the other hand, $\widehat{U}$ is generated by elements of order p since each root subgroup is isomorphic to the additive group of k. So, $\widehat{U}/[\widehat{U}, \widehat{U}]$ must be finite, and $[\widehat{U}, \widehat{U}]$ must be open in $\widehat{U}$ and hence in $\widehat{G}$. Since $\widehat{G} \supset \widehat{U}$, we have shown that $[\widehat{G}, \widehat{G}]$ is open in $\widehat{G}$.

(c) We know (by construction) that the corresponding results hold for incomplete groups, that is, U is normal in B, $B = UH$ and $H = N \cap B$. Taking the completions of both sides of the last two equalities, and using the fact that (the images of) H and N in $\widehat{G}$ are discrete, we get $\widehat{B} = \widehat{U}H$ and $H = N \cap \widehat{B}$. The normality of $\widehat{U}$ in $\widehat{B}$ is clear.

(d) The Coxeter group associated to $(\widehat{B}, N)$ is isomorphic to W, and W is irreducible since the matrix A is indecomposable.

(e) Recall that X^+ denotes the building associated with the incomplete BN-pair (B, N), and let $\mathcal{C}$ be the chamber of X^+ whose stabilizer in G is B. It follows directly from definitions that the stabilizer of $\mathcal{C}$ in $\widehat{G}$ is $\widehat{B}$. So, $Z = \bigcap_{g \in \widehat{G}} g\widehat{B}g^{-1}$ consists of elements which stabilize all chambers in X^+ (recall that G acts transitively on the set of chambers). Therefore $Z = \{1\}$. □

5. "Relative" Kac–Moody groups

Let $A = (a_{ij})_{i \in I}$ be a generalized Cartan matrix. As before, let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots. Recall that given a subset J of I, we denote by A_J the $|J| \times |J|$ matrix $(a_{ij})_{i,j \in J}$.

One can associate two *a priori* different groups to the matrix A_J: the usual Kac–Moody group $G(A_J)$, and the "relative" Kac–Moody group G_J which is defined as the subgroup of $G(A)$ generated by $\{U_{\pm\alpha_i}\}_{i \in J}$. The first result of this section asserts that these two groups are canonically isomorphic (Proposition 5.1). Next, one can consider two different topologies on $G(A_J)$ — the usual building topology and the topology induced from the building topology on $G(A)$ via the above isomorphism $G(A_J) \cong G_J$. While the two topologies may not be the same, the main result of this section is that their restrictions to the subgroup $U(A_J)$ are the same, provided A is symmetric (Theorem 5.2). We believe that Theorem 5.2 holds without the assumption that A is symmetric, but we are unable to prove it at the present time. In the next section we use Theorem 5.2 to reduce the problem of finite generation of $\widehat{U}(A)$ to a certain question about the groups $\widehat{U}(A_J)$ where J runs over all subsets of cardinality 2, once again assuming that A is symmetric.

In order to state our results precisely, we introduce the following notation. We set $G = G(A)$, $\Phi = \Phi(A)$, $W = W(A)$, $U = U(A)$ and $U^- = U^-(A)$. Fix a subset $J \subseteq I$, and let $\Pi_J = \{\alpha_i\}_{i \in J}$, let W_J be the subgroup of W generated by $\{w_i\}_{i \in J}$ and $\Phi_J = W_J(\Pi_J)$. Also let $\Phi_J^{\pm} = \Phi_J \cap \Phi^{\pm}$. Clearly, W_J (resp. Φ_J) can be canonically identified with $W(A_J)$ (resp. $\Phi(A_J)$).

We have already defined the group G_J. Note that by relations (R3), $G_J \supset U_\alpha$ for every $\alpha \in \Phi_J$. Also introduce the subgroups

$$U_J := \{U_\alpha \mid \alpha \in \Phi_J^+\} \quad \text{and} \quad U_J^- := \{U_\alpha \mid \alpha \in \Phi_J^-\}$$

of G_J. We will see shortly that $U_J = G_J \cap U$ and $U_J^- = G_J \cap U^-$.

Proposition 5.1. *There is a natural isomorphism $\varphi : G(A_J) \to G_J$. Moreover, φ maps $U(A_J)$ onto U_J and $B(A_J)$ onto B_J.*

In view of this proposition, we can identify U_J (resp. G_J) with $U(A_J)$ (resp. $G(A_J)$). Let $\widehat{U}_J$ be the closure of $U(A_J)$ in $\widehat{G}(A_J)$, as before, and let $\overline{U}_J$ be the closure of U_J in $\widehat{G}$.

Theorem 5.2. *Assume that A is symmetric. The groups $\widehat{U}_J$ and $\overline{U}_J$ are (topologically) isomorphic.*

Proof of Proposition 5.1. As before, identify $\Phi(A_J)$ with the subset Φ_J of Φ. To distinguish between generators of $G(A_J)$ and $G(A)$, we use the symbols $\{x_\alpha(u) \mid \alpha \in \Phi(A_J), u \in k\}$ for the generators of $G(A_J)$ (the generators of $G(A)$ are denoted $\{\chi_\alpha(u)\}$ as usual). From the defining presentation of Kac–Moody groups, it is clear that there exists a map $\varphi : G(A_J) \to G$ such that $\varphi(x_\alpha(u)) = \chi_\alpha(u)$ for $\alpha \in \Phi(A_J)$, $u \in k$. Clearly, $\varphi(G(A_J)) = G_J$, $\varphi(U(A_J)) = U_J$ and $\varphi(B(A_J)) = B_J$, so we only need to show that φ is injective. We proceed in several steps.

Step 1: *φ is injective on $U(A_J)$.* Let $\psi : U(A) \to U(A_J)$ be the unique homomorphism such that

$$\psi(\chi_\alpha(u)) = \begin{cases} x_\alpha(u) & \text{if } \alpha \in \Phi_J, \\ 1 & \text{if } \alpha \notin \Phi_J. \end{cases}$$

The fact that such a homomorphism exists follows immediately from Theorem 2.3. It is also clear that $\psi\varphi(g) = g$ for any $g \in U(A_J)$, whence the restriction of φ to $U(A_J)$ must be injective.

Step 2: *φ is injective on $H(A_J)$.* This follows directly from the fact that relations (R6)–(R7) are defining relations for the tori $H(A_J)$ and $H(A)$.

Step 3: *$\varphi^{-1}(B_J) = B(A_J)$ where $B_J = B \cap G_J$.* It is clear that $\varphi^{-1}(B_J) \supseteq B(A_J)$. Suppose that $\varphi^{-1}(B_J)$ is strictly larger than $B(A_J)$. Since $(B(A_J), N(A_J))$ is a BN-pair of $G(A_J)$, we conclude that $\varphi^{-1}(B_J)$ is of the form $B(A_J)W(A_K)B(A_J)$ for some non-empty subset $K \subseteq J$. This would mean that $B_J = \varphi(B(A_J)W(A_K)B(A_J))$ contains at least one of the generators of W, which is impossible since $B \cap N = H$.

Step 4: *Conclusion.* Let $K = \operatorname{Ker}\varphi$. Note that $K \subset B(A_J)$ since $B(A_J)$ is the full preimage of B_J under φ. Take any $g \in K$ and write it as $g = uh$ where $u \in U(A_J)$ and $h \in H(A_J)$. Then $\varphi(u) = \varphi(h)^{-1}$. On the other hand, it is clear that $\varphi(u) \in U$ and $\varphi(h) \in H$. Since $H \cap U = 1$ and φ is injective on both $U(A_J)$ and $H(A_J)$, we conclude that $u = h = 1$. Therefore, K is trivial. □

Before proving Theorem 5.2, we state two lemmas, which will be established at the end of the section.

Lemma 5.3. *Let $\Psi_J = \{\beta \in \Phi^+ : \langle \beta, \alpha_j^\vee \rangle \leq 0 \text{ for any } j \in J\}$. The following hold:*

(a) *For any $\gamma \in \Phi^+ \setminus \Phi_J^+$ there exist $w \in W_J$ and $\beta \in \Psi_J$ such that $\gamma = w\beta$.*

(b) *Assume that A is symmetric. Then for any $\gamma \in \Psi_J$ and $\beta \in \Phi_J^+$, the root groups $U_{-\gamma}$ and U_β commute (elementwise).*

Lemma 5.4. *Let C be a generalized Cartan matrix. Then the building topology on $U(C)$ is given by the subbase $\{gU(C)g^{-1} \cap U(C)\}_{g \in G(C)}$.*

Proof of Theorem 5.2. By definition, $\overline{U}_J$ and $\widehat{U}_J$ are the completions of U_J with respect to the topologies $(\mathcal{T}_1)$ and $(\mathcal{T}_2)$, respectively, where $(\mathcal{T}_1)$ is given by the subbase $\{gBg^{-1} \cap U_J\}_{g \in G}$ and $(\mathcal{T}_2)$ is given by the subbase $\{gB_Jg^{-1} \cap U_J\}_{g \in G_J}$. We have to show that $(\mathcal{T}_1)$ and $(\mathcal{T}_2)$ coincide.

The inequality $(\mathcal{T}_1) \geq (\mathcal{T}_2)$ is clear. Indeed, for any $g \in G_J$ we have $gBg^{-1} \cap U_J = gB_Jg^{-1} \cap U_J$ since $gB_Jg^{-1} = g(B \cap G_J)g^{-1} = gBg^{-1} \cap G_J$.

Now we prove the reverse inequality. In view of the natural isomorphism $U(A_J) \cong U_J$, Lemma 5.4 applied with $C = A_J$ reduces the proof of the inequality $(\mathcal{T}_2) \geq (\mathcal{T}_1)$ to the following statement:

Claim 5.5. *Given $g \in G$, there exists a finite set $T \subset G_J$ such that*

$$gBg^{-1} \cap U_J \supseteq \bigcap_{t \in T} t\, U_J\, t^{-1} \cap U_J.$$

Fix $g \in G$. By the Birkhoff decomposition, $g = g_- w g_+$ for some $g_+ \in B$, $w \in W$ and $g_- \in U^-$. We will show that there exist $g_1, \ldots, g_k \in G_J$ and $v \in W_J$ such that

(a) if $x \in U_J$ is such that $g_i^{-1} x g_i \in U_J$ for $1 \leq i \leq k$, then $g_-^{-1} x g_- = g_k^{-1} x g_k$ (in particular, $g_-^{-1} x g_- \in U_J$),
(b) if $y \in U_J$ is such that $v^{-1} y v \in U_J$, then $w^{-1} y w \in U$.

First, let us see why (a) and (b) will imply Claim 5.5. Indeed, let $\{g_i\}$ and v be as above, and set $T = \{g_i\}_{i=1}^k \cup \{g_k v\}$. Let $x \in \bigcap_{t \in T} t\, U_J\, t^{-1} \cap U_J$. Then by (a), $g_-^{-1} x g_- = g_k^{-1} x g_k$. Now let $y = g_-^{-1} x g_-$. Then $v^{-1} y v = (g_k v)^{-1} x (g_k v) \in U_J$ by the choice of T, so applying (b) we get that $w^{-1} g_-^{-1} x g_- w \in U$. Finally, $g^{-1} x g = g_+^{-1} w^{-1} g_-^{-1} x g_- w g_+ \in U$ since $g_+ \in B$ and B normalizes U. So, $x \in gUg^{-1}$.

Proof of (a). By definition of U^-, we can write g_- in the form $t_1 \ldots t_s$ such that each t_i lies in $U_{-\gamma_i}$ for some $\gamma_i \in \Phi^+$. By Lemma 5.3(a), for each i such that $\gamma_i \notin \Phi_J$, there exists $v_i \in W_J$ and $\beta_i' \in \Psi_J$ such that $\gamma_i = v_i \beta_i'$. Then $t_i \in U_{-\gamma_i} = U_{v_i(-\beta_i')} = v_i U_{-\beta_i'} v_i^{-1}$. Thus, we can write $g_- = x_1 \ldots x_k$ (with $k \leq 3s$), where for each $1 \leq i \leq k$ either $x_i \in U_{-\beta_i}$ with $\beta_i \in \Psi_J$ or $\beta_i \in \Phi_J$, or $x_i \in W_J$. Note that for each $1 \leq i \leq k$ either $x_i \in U_{-\beta_i}$ with $\beta_i \in \Psi_J$, or $x_i \in G_J$.

Now define $y_0, \ldots, y_k \in G$ and $g_0, \ldots, g_k \in G_J$ inductively: $y_0 = 1$; $y_i = y_{i-1} x_i$ for $1 \leq i \leq k$; $g_0 = 1$, and for $1 \leq i \leq k$ we set $g_i = g_{i-1} x_i$ if $x_i \in G_J$ and $g_i = g_{i-1}$ if $x_i \notin G_J$. Note that $y_k = g_-$ and $g_i \in G_J$ for each i.

Suppose that x satisfies the hypotheses of (a). We shall prove by induction that $y_i^{-1} x y_i = g_i^{-1} x g_i$ for all i. The base case $i = 0$ is trivial. Now let $i > 0$, and suppose that $y_{i-1}^{-1} x y_{i-1} = g_{i-1}^{-1} x g_{i-1}$. If $x_i \in G_J$, then by construction $y_{i-1}^{-1} y_i = g_{i-1}^{-1} g_i = x_i$, so the induction step is clear.

If $x_i \notin G_J$, then by construction $x_i \in U_{-\beta_i}$ for some $\beta_i \in \Psi_J$. By Lemma 5.3(b), $U_{-\beta_i}$ centralizes U_J, since U_J is generated by $\{U_\gamma\}_{\gamma \in \Phi_J^+}$. By the induction hypotheses, $y_{i-1}^{-1} x y_{i-1} = g_{i-1}^{-1} x g_{i-1} \in U_J$. Therefore,

$$y_i^{-1} x y_i = x_i^{-1} (y_{i-1}^{-1} x y_{i-1}) x_i = y_{i-1}^{-1} x y_{i-1} = g_{i-1}^{-1} x g_{i-1} = g_i^{-1} x g_i.$$

This completes the induction step and hence the proof of (a).

Proof of (b). The following argument was suggested to us (in a slightly different form) by Bertrand Remy and Pierre-Emmanuel Caprace. First of all, the statement of (b) is clearly implied by the following: for any $w \in W$ there exists $v \in W_J$ such that $wvU_J(wv)^{-1} \subseteq U$. We know that U_J is generated by the root subgroups $\{U_\alpha\}_{\alpha \in \Phi_J^+}$. Since $zU_\alpha z^{-1} = U_{z\alpha}$ for every $z \in W$, it suffices to prove the following statement about root systems:

$$\text{For any } w \in W \text{ there exists } v \in W_J \text{ such that } wv(\alpha_i) > 0 \quad \text{for all } i \in J.$$

Recall that the simple roots $\{\alpha_i\}$ are linearly independent elements of the $\mathbb{Q}$-vector space $\mathfrak{h}^*$. Let $\mathfrak{h}^*_{\mathbb{R}} = \mathfrak{h}^* \otimes_{\mathbb{Q}} \mathbb{R}$. Let $R = \{\beta \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \beta, \alpha_i^\vee \rangle > 0 \text{ for all } i\}$ be the fundamental chamber and $R_J = \{\beta \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \beta, \alpha_i^\vee \rangle > 0 \text{ for all } i \in J\}$ be the J-relative fundamental chamber (in the sense of [11, 5.7]). Now pick any $x \in R$ (it is well known that R is non-empty). By [11, 5.7, Proposition 5(i)], the union of W-translates of R is contained in the union of W_J-translates of R_J. Therefore, given $w \in W$, there exists $v \in W_J$ such that $v^{-1} w^{-1} x \in R_J$. For any $i \in J$ we have $\langle v^{-1} w^{-1} x, \alpha_i^\vee \rangle > 0$, whence $\langle x, wv\alpha_i^\vee \rangle > 0$. Since $x \in R$, the last inequality implies that $wv\alpha_i^\vee$ must be a positive coroot, whence $wv\alpha_i$ is a positive root. The proof is complete. □

Proof of Lemma 5.3. (a) This result is probably well known and follows easily from [11, 5.7, Proposition 5], but since a direct proof is very short, we present it here. The proof is by induction on height(γ).

If $\gamma \in \Phi^+ \setminus \Phi_J^+$ and height$(\gamma) = 1$, then $\gamma = \alpha_i$ for some $i \notin J$, whence for any $j \in J$ we have $\langle \gamma, \alpha_j^\vee \rangle = a_{ji} \leq 0$, so $\gamma \in \Psi_J$.

Now take any $\gamma \in \Phi^+ \setminus \Phi_J^+$ and assume that Lemma 5.3(a) holds for any root $\gamma' \in \Phi^+ \setminus \Phi_J^+$ with height(γ') < height(γ). If $\langle \gamma, \alpha_j^\vee \rangle \leq 0$ for any $j \in J$, then $\gamma \in \Psi_J$. If $\langle \gamma, \alpha_j^\vee \rangle > 0$ for some $j \in J$, let $\gamma' = w_j\gamma$. Then $\gamma' = \gamma - \langle \gamma, \alpha_j^\vee \rangle \alpha_j$ is a root of smaller height than γ, and $\gamma' \in \Phi^+ \setminus \Phi_J^+$ since the set $\Phi^+ \setminus \Phi_J^+$ is W_J-invariant. Thus, there exist $w \in W_J$ and $\beta \in \Psi_J$ such that $\gamma' = w\beta$, whence $\gamma = w_j\gamma' = (w_j w)\beta$.

(b) Since A is symmetric, there exists a symmetric W-invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ such that $(\alpha, \beta) = \langle \alpha, \beta^\vee \rangle$ for any $\alpha, \beta \in \Phi$; in particular, $(\alpha, \alpha) = 2$ for any $\alpha \in \Phi$ (see [9, Chapter 2]).

Now let $\gamma \in \Psi_J$ and $\beta \in \Phi_J^+$. Then the set $\mathbb{Z}_{>0}(\beta) + \mathbb{Z}_{>0}(-\gamma)$ does not contain any real roots. Indeed, $(\gamma, \beta) \leq 0$ by definition of Ψ_J, so for any $i, j > 0$ we have $(i\beta - j\gamma, i\beta - j\gamma) = 2(i^2 + j^2 - ij(\gamma, \beta)) \geq 2(i^2 + j^2) \geq 4$. Thus the pair $\{\beta, -\gamma\}$ is prenilpotent and moreover, the corresponding root groups commute by relations (R2). □

Proof of Lemma 5.4. It will suffice to show that $gU(C)g^{-1} \cap U(C) = gB(C)g^{-1} \cap U(C)$ for any $g \in G(C)$. Thus we fix $g \in G(C)$ and $x \in U(C)$ such that $g^{-1}xg \in B(C)$. Then $g^{-1}xg = hu$ where $u \in U(C)$ and $h \in H(C)$. We need to show that $h = 1$.

Assume that $h \neq 1$. Then we can choose a matrix D such that C is a submatrix of D and h does not lie in the center of $G(D)$ (as before, we identify $G(C)$ with a subgroup of $G(D)$). The existence of such a D follows easily from defining relations (R4) — we just have to ensure that h acts non-trivially by conjugation on one of the root subgroups.

Now let $(\mathcal{T})$ be the building topology on $G(D)$. Let $\overline{U}(C)$ (resp. $\widehat{G}(D)$, $\widehat{U}(D)$) be the completion of $U(C)$ (resp. $G(D)$, $U(D)$) with respect to $(\mathcal{T})$. By Theorem 2.5(b), $\widehat{U}(D)$ is a pro-p group, hence so is $\overline{U}(C)$. Therefore, $x^{p^n} \to 1$ in $(\mathcal{T})$ as $n \to \infty$, whence $(g^{-1}xg)^{p^n} \to 1$ in $(\mathcal{T})$ as well. On the other hand, $(g^{-1}xg)^{p^n} = (hu)^{p^n} = h^{p^n}u_n$ where $u_n \in U(C)$. Since the order of h is finite, prime to p and $\overline{U}(C)$ is compact, there exists a subsequence $\{n_k\}$ such that $h^{p^{n_k}} = h$ for all k and $\{u_{n_k}\}$ converges to some element $v \in \overline{U}(C)$. Thus $hv = 1$ in $\widehat{G}(D)$. Since $\overline{U}(C)$ is pro-p, we conclude that $h = 1$ in $\widehat{G}(D)$. So, h lies in the kernel of the natural map $G(D) \to \widehat{G}(D)$ which, as we know, coincides with the center of $G(D)$. This contradicts our initial assumption. □

6. Finite generation of $\widehat{U}$: Reduction to the rank 2 case

In this section we prove that finite generation of $\widehat{U}$ is essentially determined by rank 2 subsystems of Φ, provided A is symmetric.

Definition. Let A be a generalized Cartan matrix and $\widehat{U} = \widehat{U}(A)$. We say that $\widehat{U}$ is *well behaved* if for any non-simple root $\gamma \in \Phi(A)^+$ we have $U_\gamma \subseteq [\widehat{U}, \widehat{U}]$.

By Proposition 4.6(a), if $\widehat{U}$ is well behaved then $\widehat{U}$ is topologically generated by simple root subgroups $\{U_\alpha\}_{\alpha \in \Pi(A)}$ (in particular, $\widehat{U}$ is topologically finitely generated).

For the rest of this section we fix a matrix $A = (a_{ij})_{i,j \in I}$, and we write $\widehat{U} = \widehat{U}(A)$ and $\Phi = \Phi(A)$. For a subset J of I we define Φ_J as in the previous section. Recall that $\{\alpha_i\}_{i \in I}$ are simple roots, and w_i is the reflection associated with α_i.

Theorem 6.1. *Suppose that A is symmetric, and for any subset $J \subseteq I$ of cardinality 2, the group $\widehat{U}(A_J)$ is well behaved. Then $\widehat{U}$ is also well behaved and hence (topologically) finitely generated.*

Theorem 6.1 is an easy consequence of the following lemma.

Lemma 6.2. *Let $\gamma \in \Phi^+$ be a non-simple root. There exist simple roots α_i, α_j and $w \in W$ such that $w\alpha_i > 0$, $w\alpha_j > 0$ and $\gamma = w\alpha$ for some non-simple root $\alpha \in \Phi^+_{\{i,j\}}$.*

Proof of Lemma 6.2. If γ lies in a subsystem generated by two simple roots, that is, $\gamma = n\alpha_i + m\alpha_j$ for some $n, m \in \mathbb{N}$ and $i, j \in I$, the assertion is obvious (we can take $w = 1$). From now on assume that $\gamma = \sum_{i=1}^{l} n_i\alpha_i$ where at least three n_i's are nonzero. We will prove the lemma by induction on height(γ).

We note that there exists $k \in I$ such that height$(w_k\gamma)$ < height(γ). Indeed, if height$(w_i\gamma) \geq$ height(γ) for all i, then $\langle \gamma, \alpha_i^\vee \rangle \leq 0$ for all i and hence $\langle \alpha_i, \gamma^\vee \rangle \leq 0$ for all i. But γ is a linear combination of α_i with nonnegative coefficients, so we must have $\langle \gamma, \gamma^\vee \rangle \leq 0$. The latter is impossible since $\langle \gamma, \gamma^\vee \rangle = 2$.

Since γ does not lie in a rank two subsystem, $w_k\gamma$ is not simple. By induction, there exist simple roots α_i and α_j, $w \in W$ and $\alpha \in \Phi^+_{\{i,j\}}$ such that $w\alpha_i > 0$, $w\alpha_j > 0$ and $w_k\gamma = w\alpha$.

Note that $\gamma = w_k w\alpha$. If both $w_k w\alpha_i$ and $w_k w\alpha_j$ are positive, we are done. Suppose, this is not the case. Then we must have $w\alpha_i = \alpha_k$ or $w\alpha_j = \alpha_k$, and without loss of generality we assume that $w\alpha_i = \alpha_k$. We have

$$\begin{aligned}\gamma &= w_k(w\alpha) = w\alpha - \langle w\alpha, \alpha_k^\vee \rangle \alpha_k = w\alpha - \langle w\alpha, (w\alpha_i)^\vee \rangle w\alpha_i \\ &= w\alpha - \langle w\alpha, w(\alpha_i^\vee) \rangle w\alpha_i = w\alpha - \langle \alpha, \alpha_i^\vee \rangle w\alpha_i = w(\alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i) = w(w_i\alpha).\end{aligned}$$

Since $\alpha \in \Phi^+_{\{i,j\}}$ and α is not simple, we have $w_i\alpha \in \Phi^+_{\{i,j\}}$. The proof will be complete if we show that $w_i\alpha$ is not simple. If $w_i\alpha$ is simple, then height$(w_i\alpha)$ < height(α), so we must have $\langle \alpha, \alpha_i^\vee \rangle > 0$. Therefore, $\langle w\alpha, \alpha_k^\vee \rangle = \langle w\alpha, w\alpha_i^\vee \rangle > 0$, whence

$$\text{height}(\gamma) = \text{height}(w_k w\alpha) < \text{height}(w\alpha) = \text{height}(w_k\gamma),$$

contrary to our assumptions. □

Proof of Theorem 6.1. Fix a non-simple positive real root γ. We have to show that $U_\gamma \in [\widehat{U}, \widehat{U}]$. By Lemma 6.2, there exist $\alpha_i, \alpha_j \in \Pi$, $w \in W$ and a non-simple root $\alpha \in \Phi^+_{\{i,j\}}$ such that $w\alpha_i > 0$, $w\alpha_j > 0$ and $\gamma = w\alpha$. Let $J = \{i, j\}$ and $V = \widehat{U}_J$. Since $V \cong \widehat{U}(A_J)$ by Theorem 5.2, the hypotheses of Theorem 6.1 imply that V is topologically generated by $U_{\alpha_i} \cup U_{\alpha_j}$. Since $wU_\beta w^{-1} = U_{w\beta}$ for any $\beta \in \Phi$, we conclude that $wVw^{-1} \subseteq \widehat{U}$. By hypotheses, we also know that $U_\alpha \subset [V, V]$. Therefore, $U_\gamma = U_{w\alpha} = wU_\alpha w^{-1} \subset [wVw^{-1}, wVw^{-1}] \subseteq [\widehat{U}, \widehat{U}]$. □

In the next section we establish a sufficient condition for finite generation of $\widehat{U}$ in the rank two case:

Theorem 6.3. *Let C be a 2×2 generalized Cartan matrix. Assume that either C is finite and $|k| > 3$, or that C is affine, $|k| > 3$ and $p > 2$. Then the group $\widehat{U}(C)$ is well behaved.*

Before proving Theorem 6.3 we explain how to deduce Theorem 1.1(a). Theorem 6.4 is a direct consequence of Theorems 6.1 and 6.3. Theorem 1.1(a) is obtained by combining Theorems 6.4 and 4.1.

Theorem 6.4. *Let A be an indecomposable generalized Cartan matrix. Suppose that A is symmetric and any 2×2 submatrix of A is of finite or affine type, $|k| > 3$ and $p = \mathrm{char}(k) > 2$. Then the group $\widehat{U}(A)$ is finitely generated.*

Remark. As one can see from the proofs of Theorems 6.1 and 6.4, the assumption 'A is symmetric' was only needed to apply Theorem 5.2. We believe that Theorem 5.2 can be proved (by a completely different method) without the assumption 'A is symmetric', but under the assumption 'A_J is of affine type', using explicit realization of affine Kac–Moody groups and solution to the congruence subgroup problem. If the latter is achieved, one would be able to eliminate the assumption 'A is symmetric' from the statements of Theorems 6.1, 6.4 and 1.1(a).

7. Finite generation: The rank 2 case

In this section we prove Theorem 6.3. In order to finish the proof of Theorem 1.1(a), it would have been enough to prove Theorem 6.3 for symmetric matrices of finite or affine type. However, we decided to prove the full version of Theorem 6.3, including the computationally demanding case of non-symmetric affine matrices, because it may have applications to other problems, including possible generalization of Theorem 1.1(a) (as explained at the end of Section 6).

If C is of finite type, the group $G(C)$ is finite, so $\widehat{U}(C)$ is isomorphic to $U(C)$. Theorem 6.3 in this case follows easily from defining relations of type (R2) (see [12], where the coefficients $\{C_{mn\alpha\beta}\}$ are computed explicitly in terms of the root system generated by α and β). Before considering the affine case, we make some general remarks about affine Kac–Moody groups.

A good general reference for incomplete affine Kac–Moody groups is the thesis of Ramagge [14] (see also [15,16]). In particular, using results of [14], one can obtain an explicit realization of incomplete twisted affine Kac–Moody groups. An explicit realization of complete twisted affine Kac–Moody groups is probably known, however we are unaware of a proof in the literature. For completeness, we shall demonstrate such realization in the case of 2×2 matrices — see Proposition 7.2.

Let C be an affine matrix (of arbitrary size). Then the incomplete group $G(C)$ modulo its finite center is isomorphic to the group of fixed points of a finite order (possibly trivial) automorphism ω_C of the group of $k[t, t^{-1}]$-points of some simply-connected Chevalley group. The automorphism ω_C is non-trivial if and only if C is a *twisted* affine matrix.[8] The complete group $\widehat{G}(C)$ has an analogous description where the ring $k[t, t^{-1}]$ is replaced by the field $k((t))$. Furthermore, $\widehat{G}(C)$ is isomorphic to the group of $k((t))$-points of some simple algebraic group $\mathbb{G}_C$ defined over $k((t))$: if C is non-twisted, $\mathbb{G}_C$ is the Chevalley group mentioned above; if C is twisted, $\mathbb{G}_C$ is non-split.

One may ask if every simple algebraic group over $k((t))$ is isogenous to one of the form $\widehat{G}(C)$ for some C. The answer is 'no' since the groups $\widehat{G}(C)$ are always residually split. Moreover, there is a bijective correspondence between isogeny classes of residually split simple algebraic groups over $k((t))$ and groups of the form $\widehat{G}(C)$, with C affine. If C is an affine matrix of type $X_n^{(r)}$ (in the notation of [9, Chapter 4]), then $\widehat{G}(C)$ is a residually split group whose isogeny class is given by Tits' index of the form ${}^rX^{(d)}_{n,m}$ for some m, d in the notation of [23] (these conditions determine the isogeny class uniquely).

Now assume that C is a 2×2 affine matrix. Up to isomorphism of the corresponding Kac–Moody groups, there are only two possibilities:

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \text{ (non-twisted case)}, \quad \text{and} \quad C = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \text{ (twisted case)}.$$

In the first case C has type $A_1^{(1)}$, the Tits index is ${}^1A^{(1)}_{1,1}$, and $\widehat{G}(C)$ is isomorphic to $PSL_2(k((t)))$. In the second case C has type $A_2^{(2)}$, the Tits index is ${}^2A^{(1)}_{2,1}$, and $\widehat{G}(C)$ is isomorphic to $SU_3(k((t)), h, \sigma)$, where σ is a k-automorphism of $k((t))$ of order 2 and h is a hermitian form in three variables relative to σ (see [23]). If $p \neq 2$, we can (and will) assume that

$$\sigma(t) = -t \quad \text{and} \quad h((x_1, x_2, x_3), (y_1, y_2, y_3)) = x_1\sigma(y_3) - x_2\sigma(y_2) + x_3\sigma(y_1).$$

[8] For a classification of twisted affine matrices see [9, Chapter 4].

Then

$$SU_3(k((t)), h, \sigma) = \{g \in SL_3(k((t))) \mid J^{-1}g^*J = g^{-1}\} \quad \text{where } J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $g \mapsto g^*$ is the map from $SL_3(k((t)))$ to $SL_3(k((t)))$ obtained by applying σ to each entry of g followed by transposition.

The proof of Theorem 6.3 in both cases follows the same strategy, but the twisted case requires more computations. Our starting point is the following obvious lemma:

Lemma 7.1. *Let P be a profinite group, and let $P_1 \supset P_2 \supset \cdots$ be a descending chain of open normal subgroups of P which form a base of neighborhoods of identity. For each $i \in \mathbb{N}$ choose elements $g_{i,1}, \ldots, g_{i,n_i} \in P_i$ which generate P_i modulo P_{i+1}. Let K be a closed subgroup of P such that $g_{i,j} \in KP_{i+1}$ for all i and j. Then K contains P_1.* □

We shall apply this lemma with $P = \widehat{U}, K = [P, P]$ and a certain filtration $\{P_i\}$ of P satisfying the above conditions and such that P_1 contains U_γ for every non-simple positive root γ. Clearly, this will prove that $\widehat{U}$ is well behaved, so we only need to show the existence of a filtration with required properties.

Case 1: $C = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$.

In this case, $\Phi^+ = \{(n+1)\alpha_1 + n\alpha_2, n\alpha_1 + (n+1)\alpha_2\}_{n \in \mathbb{Z}_{\geq 0}}$. The group $\widehat{U}$ can be embedded into $SL_2(k[[t]])$ via the following map:

$$\chi_{(n+1)\alpha_1+n\alpha_2}(u) \mapsto e_n(u) := \begin{pmatrix} 1 & ut^n \\ 0 & 1 \end{pmatrix}, \qquad \chi_{n\alpha_1+(n+1)\alpha_2}(u) \mapsto f_{n+1}(u) := \begin{pmatrix} 1 & 0 \\ ut^{n+1} & 1 \end{pmatrix},$$

for all $n \in \mathbb{Z}_{\geq 0}$ and $u \in k$. For each $n \geq 0$ let $E_n = \{e_n(u) \mid u \in k\}$ and $F_n = \{f_n(u) \mid u \in k\}$. We also define the elements $\{h_n(u) \mid n \geq 1, u \in k\}$ by

$$h_n(u) = \begin{pmatrix} 1 + ut^n & 0 \\ 0 & (1 + ut^n)^{-1} \end{pmatrix}.$$

Under the above identification, $\widehat{U}$ consists of matrices in $SL_2(k[[t]])$ whose reduction mod t is upper-unitriangular. Now define the filtration $\{P_n\}$ as follows: $P_1 = P_2E_1$, and for $n \geq 2$ set

$$P_n = SL_2^n(k[[t]]) = \{g \in SL_2(k[[t]]) \mid g \equiv 1 \bmod t^n\}.$$

Clearly, P_1 contains all non-simple positive root subgroups.

For each $n \geq 2$, P_n is generated modulo P_{n+1} by the elements $\{e_n(u), f_n(u), h_n(u) \mid u \in k\}$, and P_1 is generated modulo P_2 by $\{e_1(u)\}$. Direct computation shows that $e_n(u) \equiv [h_1(1), e_{n-1}(u/2)] \bmod P_{n+1}$ for $n \geq 1, f_n(u) \equiv [h_1(1), f_{n-1}(-u/2)] \bmod P_{n+1}$ for $n \geq 2$ and $h_n(u) \equiv [e_1(1), f_{n-1}(u)] \bmod P_{n+1}$ for $n \geq 2$. So, all the hypotheses of Lemma 7.1 are satisfied, and we are done with Case 1.

Case 2: $C = \left(\begin{smallmatrix} 2 & -4 \\ -1 & 2 \end{smallmatrix}\right)$.

Let $\delta = 2\alpha_1 + \alpha_2$. Then $\Phi^+ = \{\pm\alpha_1 + n\delta \mid n \in \mathbb{Z}\} \cup \{\pm 2\alpha_1 + (2n+1)\delta \mid n \in \mathbb{Z}\}$ (see [9, Exercise 6.6]). For each $\alpha \in \Phi$ define an element $e_\alpha \in \mathfrak{sl}_3(k[t, t^{-1}])$ as follows:

α	e_α
$\alpha_1 + 2n\delta$	$(e_{12} + e_{23})t^{2n}$
$\alpha_1 + (2n+1)\delta$	$2(e_{12} - e_{23})t^{2n+1}$
$2\alpha_1 + (2n+1)\delta$	$e_{13}t^{2n+1}$
$-\alpha_1 + 2n\delta$	$2(e_{21} + e_{32})t^{2n}$
$-\alpha_1 + (2n+1)\delta$	$(e_{21} - e_{32})t^{2n+1}$
$-2\alpha_1 + (2n+1)\delta$	$e_{31}t^{2n+1}$

Proposition 7.2. *Let $\widehat{G} = \widehat{G}(C)$, $\widehat{U} = \widehat{U}(C)$ and $\mathcal{G} = \{g \in SL_3(k((t))) \mid J^{-1}g^*J = g^{-1}\}$. Then $\widehat{G}$ is isomorphic to $\mathcal{G}$ (as a topological group) via the map ι defined by*

$$\iota : \chi_\alpha(u) \mapsto 1 + (ue_\alpha) + (ue_\alpha)^2/2 \quad \text{for } \alpha \in \Phi \text{ and } u \in k.$$

Furthermore, $\iota(\widehat{U}) = \mathcal{U}$ where $\mathcal{U} = \{g \in \mathcal{G} \cap SL_3(k[[t]]) \mid g \text{ is upper-unitriangular} \bmod t\}$.

Remark. The expression $1 + (ue_\alpha) + (ue_\alpha)^2/2$ should really be "thought of" as $\exp(ue_\alpha)$ since $e_\alpha^3 = 0$ for every $\alpha \in \Phi$.

The proof of Proposition 7.2 will be given in Appendix. Henceforth we identify $\widehat{G}$ with $\mathcal{G}$ and $\widehat{U}$ with $\mathcal{U}$. Before proceeding, we introduce some terminology. Let $M_3(k)$ denote the space of 3×3 matrices over k.

Definition. Let $g \in GL_3^1(k[[t]])$. Write g in the form $1 + \sum_{i\geq 1} g_i t^i$ where $g_i \in M_3(k)$, and let n be the smallest integer such that $g_n \neq 0$. We will say that g has *degree* n and write $\deg(g) = n$. The matrix g_n will be called the *leading coefficient* of g; we will write $\mathrm{LC}(g) = g_n$.

Given a subgroup H of $GL_3^1(k[[t]])$ and $n \geq 1$, let

$$L_n(H) = \{\mathrm{LC}(g) \mid g \in H \text{ and } \deg(g) = n\} \cup \{0\}.$$

Then it is easy to see that $L_n(H)$ is an $\mathbb{F}_p$-subspace of $M_3(k)$. The following result is also straightforward.

Lemma 7.3. *Let S be a subgroup of $GL_3^1(k[[t]])$. For each $i \geq 1$ let $S_i = S \cap GL_3^i(k[[t]])$. Fix $n \in \mathbb{N}$, and let $X \subset S$ be a set of elements of degree n. Then X generates S_n modulo S_{n+1} if and only if the set $\{\mathrm{LC}(g) \mid g \in X\}$ spans $L_n(S)$.* □

Now we return to the proof of Theorem 6.3. Let $P = \widehat{U}$. Define the filtration $\{P_n\}_{n=1}^{\infty}$ of P as follows: if $n \geq 2$, set $P_n = P \cap GL_3^n(k[[t]])$, and let P_1 be the set of matrices in $P \cap GL_3^1(k[[t]])$ whose $(3, 1)$-entry lies in $t^2k[[t]]$.

Consider the following elements of $\widehat{G}$:

$$\{e_n^{(1)}(u), f_n^{(1)}(u), h_n(u) \mid n \in \mathbb{Z}, u \in k\} \quad \text{and} \quad \{e_n^{(2)}(u), f_n^{(2)}(u) \mid n \text{ is odd}, u \in k\},$$

where

$$e_n^{(1)}(u) = \chi_{\alpha_1+n\delta}(u), \qquad f_n^{(1)}(u) = \chi_{-\alpha_1+n\delta}(u), \quad \text{and} \quad h_n(u) = [e_0^{(1)}(u), f_n^{(1)}(1)]$$
$$e_n^{(2)}(u) = \chi_{2\alpha_1+n\delta}(u), \qquad f_n^{(2)}(u) = \chi_{-2\alpha_1+n\delta}(u).$$

Let $E_n^{(i)}$, $F_n^{(i)}$ and H_n be the subsets $\{e_n^{(i)}(u)\}$, $\{f_n^{(i)}(u)\}$ and $\{h_n(u)\}$, respectively.

Now consider the subsets $\{Z_n\}_{n=1}^{\infty}$ of P_1 defined as follows:

$$Z_1 = E_1^{(2)} \cup E_1^{(1)} \cup H_1 \cup F_1^{(1)}, \qquad Z_{2n} = E_{2n}^{(1)} \cup F_{2n}^{(1)} \cup H_{2n}, \quad \text{and}$$
$$Z_{2n+1} = E_{2n+1}^{(1)} \cup F_{2n+1}^{(1)} \cup H_{2n+1} \cup E_{2n+1}^{(2)} \cup F_{2n+1}^{(2)} \quad \text{for } n \geq 1.$$

We claim that Z_n generates P_n modulo P_{n+1} for each $n \geq 1$. This result follows directly from Lemma 7.3 applied with $S = P_1$ and $\{S_n\} = \{P_n\}$. Indeed, for each $n \geq 1$, define $L_n \subset M_3(k)$ as follows:

$$L_n = \{g \in M_3(k) \mid g^t J = (-1)^n J g\} \quad \text{where } g^t \text{ is the transposed of } g.$$

It is clear from the definitions that $L_n(P_1) \subseteq L_n(\widehat{U}) = L_n$. On the other hand, direct computation shows that for $n \geq 2$, all (non-identity) elements of Z_n have degree n and their leading coefficients span L_n, so $L_n(P_1) = L_n$. Similarly, one shows that the leading coefficients of elements of Z_1 span $L_1(P_1)$.

In order to finish the proof by using Lemma 7.1, we need suitable commutation relations between the elements $\{e_n^{(i)}(u), f_n^{(i)}(u), h_n(u)\}$. Once again, these are obtained by direct computation:

$$e_{2n}^{(1)}(u) \equiv [e_1^{(2)}(1), f_{2n-1}^{(1)}(-u)] \mod P_{2n+1} \qquad e_{2n+1}^{(1)}(u) \equiv [e_1^{(2)}(1), f_{2n}^{(1)}(u)] \mod P_{2n+2}$$
$$f_{2n}^{(1)}(u) \equiv [f_1^{(2)}(1), e_{2n-1}^{(1)}(u)] \mod P_{2n+1} \qquad f_{2n+1}^{(1)}(u) \equiv [f_1^{(2)}(1), e_{2n}^{(1)}(-u)] \mod P_{2n+2}$$
$$e_{2n+1}^{(2)}(u) \equiv [e_0^{(1)}(1), e_{2n+1}^{(1)}(-u/4)] \mod P_{2n+2} \qquad f_{2n+1}^{(2)}(u) \equiv [f_1^{(1)}(1), f_{2n}^{(1)}(-u/4)] \mod P_{2n+2}.$$

Finally, elements $\{h_n(u)\}_{n\geq 1}$ lie in $[P, P]$ by definition.

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Appendix. On explicit realization of twisted affine Kac–Moody groups

In this section we prove Proposition 7.2. Recall that

$$C = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1}$$

Proposition 7.2. *Let $\widehat{G} = \widehat{G}(C)$, $\widehat{U} = \widehat{U}(C)$ and $\mathcal{G} = \{g \in SL_3(k((t))) \mid J^{-1}g^*J = g^{-1}\}$. Then $\widehat{G}$ is isomorphic to $\mathcal{G}$ (as a topological group) via the map ι defined by*

$$\iota : \chi_\alpha(u) \mapsto 1 + (ue_\alpha) + (ue_\alpha)^2/2 \quad \text{for } \alpha \in \Phi \text{ and } u \in k. \tag{2}$$

Furthermore, $\iota(\widehat{U}) = \mathcal{U}$ where $\mathcal{U} = \{g \in \mathcal{G} \cap SL_3(k[[t]]) \mid g \text{ is upper-unitriangular mod } t\}$.

Proof. We proceed in several steps.

Step 1: First, we claim that (2) defines a (unique) homomorphism ι_0 from the incomplete group $G = G(C)$ to $SL_3(k[t, t^{-1}])$. This follows directly from the presentation of G by generators and relators.

Step 2: Let $K = \operatorname{Ker} \iota_0$ and let Z be the kernel of the natural map $G \to \widehat{G}$. At this step we show that $K \subseteq Z$. Recall that $(\mathcal{T}_{\text{build}})$ denotes the building topology on G. Let $(\mathcal{T}_{aux})$ be the topology on $\iota_0(G)$ given by the base $\{\iota(V)\}$ where V runs over subgroups of G open in $(\mathcal{T}_{\text{build}})$. Let $\widetilde{G}$ be the completion of $\iota_0(G)$ with respect to $(\mathcal{T}_{aux})$. Clearly, there exists a continuous homomorphism $\varepsilon : \widehat{G} \to \widetilde{G}$. By [18, Theorem 2.A.1], $\widehat{G}$ is topologically simple, whence ε is injective (since $\widetilde{G}$ is clearly non-trivial). Thus we conclude that $K \subseteq Z$. It follows immediately that the map $\iota_0 : G \to \iota_0(G)$ canonically extends to an isomorphism of topological groups $\iota : \widehat{G} \to \widetilde{G}$.

Step 3: Consider the topology $(\mathcal{T}_{\text{cong}})$ on $\iota_0(G)$ induced from the congruence topology on $SL_3(k((t)))$, and let $\overline{G}$ be the completion of $\iota_0(G)$ with respect to $(\mathcal{T}_{\text{cong}})$. At this step we show that $\overline{G}$ coincides with $\mathcal{G}$. It is clear that $\overline{G} \subseteq \mathcal{G}$ since $\iota_0(G) \subset \mathcal{G}$ by construction and $\mathcal{G}$ is closed in the congruence topology.

Now we prove the reverse inclusion $\mathcal{G} \subseteq \overline{G}$. The group $\mathcal{G}$ is an isotropic simple algebraic group over the local field $k((t))$ and hence has a BN-pair $(\mathcal{B}, \mathcal{N})$ given by Bruhat–Tits theory. An explicit description of $(\mathcal{B}, \mathcal{N})$ is given in [23, 1.15]: $\mathcal{B} = \{g \in \mathcal{G} \cap SL_3(k[[t]]) \mid g \text{ is upper-triangular mod } t\}$ and $\mathcal{N}$ is the semi-direct product of the group

$$D := \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma(x)^{-1} \end{pmatrix} \middle| \, x \in k((t)) \right\}$$

and the group of order 2 generated by the matrix J defined previously in (1). Note that $\mathcal{B} = \iota(H)\mathcal{U}$ where H is the diagonal subgroup of G.

Let $\overline{U}$ be the closure of $\iota_0(U)$ in $\mathcal{G}$. Applying Lemma 7.1 with $P = \mathcal{U}$ and $\{P_i\} = \{P \cap SL_3^i(k[[t]])\}$ and arguing as in Section 7, we conclude that $\overline{U} = \mathcal{U}$. From the explicit description of $\mathcal{N}$, it is clear that $\mathcal{N} = (\mathcal{N} \cap \mathcal{U})\iota_0(N)$. Thus $\overline{G}$ contains both $\mathcal{U}$ and $\mathcal{N}$. Since $(\mathcal{B}, \mathcal{N})$ is a BN-pair and $\mathcal{B} \subset \iota_0(N)\mathcal{U}$, it follows that $\mathcal{G}$ is generated by $\mathcal{U}$ and $\iota_0(N)$. Since $\overline{G}$ contains $\overline{U} = \mathcal{U}$ and $\iota_0(N)$, we conclude that $\overline{G} = \mathcal{G}$.

Step 4: Now we prove that the groups $\overline{G}$ and $\widetilde{G}$ are topologically isomorphic. Equivalently, we will show that the topologies $(\mathcal{T}_{\text{cong}})$ and $(\mathcal{T}_{aux})$ on $\iota_0(G)$ coincide. The inequality $(\mathcal{T}_{aux}) \leq (\mathcal{T}_{\text{cong}})$ is clear. This inequality also implies that there is a natural homomorphism $\varepsilon_1 : \overline{G} \to \widetilde{G}$. Since $\overline{G} = \mathcal{G}$ is simple, ε_1 is injective.

Now suppose that $(\mathcal{T}_{\text{cong}})$ is strictly stronger than $(\mathcal{T}_{aux})$. Since both topologies $(\mathcal{T}_{\text{cong}})$ and $(\mathcal{T}_{aux})$ are countably based, it follows that there is a subgroup $V \subset \iota_0(G)$, open in $(\mathcal{T}_{\text{cong}})$, and a sequence $\{g_n\}$ in $\iota_0(G)$ such that

(a) g_n converges to 1 with respect to $(\mathcal{T}_{aux})$;
(b) $g_n \notin V$ for all n.

Condition (a) implies that $g_n \in \iota_0(B)$ for all sufficiently large n. Let $\overline{B}$ be the closure of $\iota_0(B)$ in $\overline{G}$. Clearly, $\overline{B}$ is compact and countably based, so there exists a subsequence $\{g_{n_k}\}$ which converges to some $g \in \overline{B}$. Since $\{g_{n_k}\}$ converges to 1 with respect to $(\mathcal{T}_{aux})$, it follows that g lies in the kernel of $\varepsilon_1 : \overline{G} \to \widetilde{G}$. Since ε_1 is injective, we conclude that $g = 1$, contrary to condition (b).

Step 5: Combining steps 3 and 4, we get $\widetilde{G} = \overline{G} = \mathcal{G}$, so the map $\iota : \widehat{G} \to \widetilde{G}$ defined at the end of step 2 has the desired properties. The equality $\iota(\widehat{U}) = \mathcal{U}$ holds by construction. □

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