Homework 5
Due 17/06/2008

Warm-Up Exercises

1. Prove that NL is closed under union and under intersection.[1]

2. (a) Show that PAL, the language of all palindromes over $\{0, 1\}$, can be decided using logarithmic space. What is the running time of your Turing machine?

   (b) Show that there is a non-deterministic Turing machine that decides $\overline{\mathsf{PAL}}$ in linear time and logarithmic space.

3. Show that multiplication of two square matrices over the integers can be done in deterministic space $O(\log n)$. Note that n is the total length of the input.

4. The following problem is complete for a certain complexity class; which one?
   Input: A deterministic Turing machine M that has only one Read-Write tape (on which it gets the input) and an input x for M.
   Question: Does M accept x without leaving the first $|x| + 1$ places of the tape?

Exercises for Submission

1. Consider the problem CYCLE-1 defined as follows:
   Input: A directed graph with out-degree at most one.
   Question: Does it contain a cycle?
   Show that CYCLE-1 $\in$ L.

2. In the problem UPATH the input is an *undirected* graph $G = (V, E)$, and two vertices $s, t \in V$, and the question is "Is s connected to t in G?".

   (a) Show that UPATH $\in$ NL.
   Remark: UPATH is known to be in L and is complete for a class called SL.

   (b) Show that $\overline{\text{2-Col}} \in$ NL. Hint: Use Homework 2, question (2a).

   (c) Show that $\overline{\text{2-Col}} \leq_L$ UPATH and that UPATH $\leq_L \overline{\text{2-Col}}$.

3. (a) Prove that if PH = PSPACE then the polynomial-time hierarchy collapses.

   (b) Prove that $\mathsf{PSPACE}^{\mathsf{TQBF}} = \mathsf{PSPACE}$ and $\mathsf{P}^{\mathsf{TQBF}} = \mathsf{PSPACE}$, and conclude that to prove $\mathsf{P} \neq \mathsf{PSPACE}$ one needs a non-relativizing argument. Is $\mathsf{PSPACE} = \mathsf{NP}^{\mathsf{TQBF}}$?

   (c) Prove or disprove: $\mathsf{EXP}^{\mathsf{EXPCOM}} = \mathsf{EXP}$.

4. Let us define the class NL^* just like NL, except that we allow the head of the witness tape to move in both directions. In more detail, NL^* is the class of all languages L for which there exists a Turing machine with the following criteria:
   Input tape: Read only, move in both directions.
   Witness tape: Read only, move in both directions.
   Work tape: Read-Write, move in both directions.
   The machine itself is deterministic (the guesses are the value of the witness tape). The space complexity is the size of the work tape, and is bounded by $O(\log n)$. We say the machine accepts an input if and only if there exists a setting for the witness tape, with which the machine accepts.
   Prove that $\mathsf{NP} \subseteq \mathsf{NL}^*$, and conclude that if $\mathsf{P} \neq \mathsf{NP}$ then $\mathsf{NL} \neq \mathsf{NL}^*$.

[1] That is, if $A, B \in \mathsf{NL}$ then $A \cup B \in \mathsf{NL}$ and $A \cap B \in \mathsf{NL}$.

5. (a) Show that there exists a Turing machine U that given $x, \alpha \in \{0,1\}^*$ outputs $M_\alpha(x)$ using space $s(n)$, where M_α denotes the Turing machine represented by α whose space complexity is $s(n) \geq \log n$.
   (b) Prove the space hierarchy theorem: if f, g are two functions that satisfy $\log n \leq f(n) = o(g(n))$, then $\mathsf{SPACE}(f(n)) \subsetneq \mathsf{SPACE}(g(n))$.
   Hint: Use (5a) and the proof of the time hierarchy theorem from class.
   (c) Conclude that $\mathsf{NTIME}(n^{1004}) \subsetneq \mathsf{PSPACE}$.
6. Prove that for any $c \geq 2$, $\mathsf{SPACE}(\log^c n)$ is closed under the Kleene star.[2]
7. Two vertices u, v of a directed graph G are in the same strongly connected component (SCC) if there is a directed path from u to v and there is a directed path from v to u. A directed graph is *strongly connected* if all its vertices are in the same SCC.
   (a) Prove that the following problems are in NL:
      i. Given a directed graph G and two vertices in it u, v, are u and v in the same SCC?
      ii. Given a directed graph G, does G contain at least 2008 SCCs?
      iii. Given a directed graph G, does G contain exactly 2008 SCCs?
   (b) Prove that deciding whether a directed graph is strongly connected is NL-complete.

Bonus Exercise

Let $G = (V, E)$ be a directed graph and $v_0 \in V$ be a vertex. Consider the following two-person game: We start with a token on v_0. In the even stages, Player 1 moves the token from v_{2i} to some vertex v_{2i+1} adjacent to v_{2i}, and in the odd stages, Player 2 moves the token from v_{2i+1} to some vertex v_{2i+2} adjacent to v_{2i+1}. No player may move to a vertex that has already been visited. A player wins by forcing the opponent to a position from which there is no legal next move. Prove that the problem: "Given an instance (G, v_0), decide whether Player 1 has a forced win?" is PSPACE-complete.

[2] That is, if $A \in \mathsf{SPACE}(\log^c n)$ then $A^* \in \mathsf{SPACE}(\log^c n)$.