Warm-up Exercises

Prove:

- $\mathsf{DTIME}(2^n) \subsetneq \mathsf{NTIME}(2^{2n})$.
- For any language $L \in \mathsf{NP} \cap \mathsf{coNP}$ we have $\mathsf{NP}^L = \mathsf{NP}$.
- If $\mathsf{3SAT}$ is polynomial-time reducible to $\overline{\mathsf{3SAT}}$ then $\mathsf{PH} = \mathsf{NP}$.

Exercises for Submission

1. (a) Let $\Sigma_2\mathsf{SAT}$ denote the following decision problem: given a quantified formula ψ of the form $\psi = \exists x \in \{0,1\}^n \; \forall y \in \{0,1\}^n . \; \phi(x,y) = 1$, where ϕ is a CNF formula, decide whether ψ is true. Prove that if $\mathsf{P} = \mathsf{NP}$ then $\Sigma_2\mathsf{SAT} \in \mathsf{P}$.

   (b) Prove that if $\Sigma_2\mathsf{SAT} \in \mathsf{SIZE}(n^{20})$ then the polynomial-time hierarchy collapses.

2. (a) The language $\mathsf{FIRST\text{-}ACCEPT}$ consists of those pairs (C_1, C_2) for which C_1, C_2 are Boolean circuits, and the lexicographically first string x for which $C_1(x) = 1$ is also accepted by C_2.[1] Prove that $\mathsf{FIRST\text{-}ACCEPT} \in \mathsf{P}^{\mathsf{NP}}$.
   Remark: It can be shown that $\mathsf{FIRST\text{-}ACCEPT}$ is P^{NP}-complete.

   (b) Prove that $\mathsf{NP} \cup \mathsf{coNP} \subseteq \mathsf{P}^{\mathsf{NP}} \subseteq \Sigma_2^p \cap \Pi_2^p$. Is P^{NP} closed under complement?[2]

   (c) Prove that $\mathsf{NP} = \mathsf{P}^{\mathsf{NP}}$ if and only if $\mathsf{NP} = \mathsf{coNP}$.

3. We define the class $\mathbf{S}_2^p$ as the set of all languages L for which there exist a polynomial-time Turing machine M and a polynomial p such that for all $x \in \{0,1\}^*$,

$$x \in L \Rightarrow \exists y \in \{0,1\}^{p(|x|)} \; \forall z \in \{0,1\}^{p(|x|)} . \; M(x,y,z) = 1$$
$$x \notin L \Rightarrow \exists z \in \{0,1\}^{p(|x|)} \; \forall y \in \{0,1\}^{p(|x|)} . \; M(x,y,z) = 0$$

   (a) Is $\mathbf{S}_2^p$ closed under complement?

   (b) Prove that $\mathbf{S}_2^p \subseteq \Sigma_2^p \cap \Pi_2^p$.

   (c) (Bonus) Prove a stronger form of Karp-Lipton Theorem: if $\mathsf{NP} \subseteq \mathsf{P/poly}$ then $\mathsf{PH} = \mathbf{S}_2^p$.

4. Prove that there exists an oracle O such that $\mathsf{P}^O = \mathsf{PH}^O$.

5. Prove that for any two functions f, g such that $g(n) = \omega(f(n) \log f(n))$,

$$\mathsf{NTIME}(f(n)) \neq \mathsf{coNTIME}(g(n)).$$

[1] A string x *lexicographically precedes* a string y if the first position i in which they differ has $x_i = 0$ and $y_i = 1$.
[2] A class C is *closed under complement* if $L \in C$ implies $\overline{L} \in C$, or equivalently $C = \mathsf{co}C$.