Warm-up Exercises

1. Prove that for any language L and any class of languages C, L is C-complete if and only if $\overline{L}$ is co$C$-complete.
2. For two languages L_1, L_2 define $L_1 \Delta L_2 = (L_1 \setminus L_2) \cup (L_2 \setminus L_1)$.
   We say that a class C is closed under Δ if $L_1, L_2 \in C$ implies $L_1 \Delta L_2 \in C$.
   For each class decide if it is closed under Δ (if known): P, $\mathsf{P/poly}$, NP, $\mathsf{NP} \cap \mathsf{coNP}$.
3. Prove that the following problems are self reducible by a polynomial Cook reduction from the search version to the decision version of the same problem.
   (a) $\mathsf{Clique} = \{ (G,k) \mid G$ contains a clique of size $k \}$.[1]
   (b) $\mathsf{GraphIsomorphism} = \{ (G_1, G_2) \mid G_1$ and G_2 are isomorphic $\}$.[2]

Exercises for Submission

1. Let $\mathsf{UpToOneSat}$ be the following language:
   $\mathsf{UpToOneSat} = \{ \phi \mid \phi$ is a CNF formula that has at most one satisfying assignment$\}$.
   Prove that $\mathsf{UpToOneSat} \in \mathsf{NP}$ if and only if $\mathsf{NP} = \mathsf{coNP}$.
2. We say that a non-deterministic machine is *nice* if for every input $x \in \{0,1\}^*$ the following holds: Every computation path returns either 'accept','reject' or 'quit'. There is at least one non-quit path, and all non-quit paths have the same value.
   Let NICE be the class of all languages that are accepted by some non-deterministic, polynomial time, nice machine. Prove that $\mathsf{NICE} = \mathsf{NP} \cap \mathsf{coNP}$.
3. The class DP is defined as the set of all languages L for which there are two languages $L_1 \in \mathsf{NP}$ and $L_2 \in \mathsf{coNP}$ such that $L = L_1 \cap L_2$. Define:
   $\mathsf{SAT\text{-}UNSAT} = \{ (\phi_1, \phi_2) \mid \phi_1$ and ϕ_2 are CNF formulas; ϕ_1 is satisfiable and ϕ_2 is not$\}$,
   $\mathsf{MAX\text{-}IndSet} = \{ (G,k) \mid$ the largest independent set in G has size $k\}$.
   (a) Show that $\mathsf{DP} \subseteq \Sigma_2^p$.
   (b) Show that $\mathsf{SAT\text{-}UNSAT}$ is DP-complete, i.e., $\mathsf{SAT\text{-}UNSAT} \in \mathsf{DP}$ and every language in DP is polynomial-time reducible to it.
   (c) Is $\mathsf{DP} = \mathsf{NP} \cap \mathsf{coNP}$? (prove or disprove or show that it is equivalent to a well-known open question).
   (d) (Bonus) Show that $\mathsf{MAX\text{-}IndSet}$ is DP-complete.
4. Prove that $\mathsf{P} \subseteq \mathsf{P/poly}$.
   Hint: Similar to the proof of Cook-Levin Theorem (and actually simpler).
5. Prove that if every unary NP-language is in P then $\mathsf{EXP} = \mathsf{NEXP}$, and conclude that if $\mathsf{EXP} \neq \mathsf{NEXP}$ then there exists a language $L \in \mathsf{NP} \setminus \mathsf{P}$ that is not NP-complete.
   Remark: It is known that there exists a language $L \in \mathsf{NP} \setminus \mathsf{P}$ that is not NP-complete assuming the weaker assumption $\mathsf{P} \neq \mathsf{NP}$ (Ladner's Theorem).

[1] The decision version is "Given a pair (G,k) does G contain a clique of size k?" and the search version is "Given a pair (G,k) find a clique of size k in G if exists, and reject otherwise".

[2] Two graphs are *isomorphic* if there is a way to label the vertices of one graph, such that the two graphs become identical.