Problem set 8
Computational Complexity.

1. (a) Give a self-reduction for Clique. That is, show that given an oracle access to the decision variant of Clique, one can solve the search variant of Clique (find a Clique of maximal size) in polynomial time.
   (b) We say that two graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ are isomorphic if there is a permutation $\pi : V_1 \to V_2$ such that $(x, y) \in E_1 \Leftrightarrow (\pi(x), \pi(y)) \in E_2$.
   Let $GISO = \{(G_1, G_2) \mid G_1 \text{ is isomorphic to } G_2\}$
   Give a self-reduction for $GISO$. That is, show how to solve the following problem in polynomial time given an oracle access to $GISO$:
   Input: two graphs G_1, G_2.
   Output: a permutations π which shows that G_1, G_2 are isomorphic, if they are, or 'NO' if they are not.

2. Let A be a *minimization* problem. The problem $Gap_{[\alpha,\beta]} - A$ is, given an input x, decide whether there exist a solution of size at most α or every solution is of size at least β.
   We say that an algorithm is a c-approximation for A if it finds a solution of size at most c times the optimal one ($c > 1$).
   Prove that, if $P \neq NP$ and $Gap_{[\alpha,\beta]} - A \in NP - hard$ then for any $c \leq \frac{\beta}{\alpha}$ there is no polynomial time c-approximation for A.

3. Disprove the following claim:
   Let A be an NP maximization problem, for which there is no polynomial time c-approximation ($c < 1$), unless $P = NP$. For every α, β so that $0 < \alpha < \beta < 1$ and $\frac{\alpha}{\beta} < c$ we have
   $$Gap_{[\alpha,\beta]} - A \in NP - hard$$

4. Prove that the $Vertex - Cover$ problem is NP-hard to approximate to within some constant $c > 1$ even when the degree of the graph is bounded by some constant d (of your choice).

5. (optional bonus question). A cut S is bad for an expander if $E(S, \overline{S}) < c|S|$. In class we saw that Pr[exists a bad cut S of size $< n/2$] $<$ some term that goes to zero.
   (a) Show that Pr[exists a bad cut S of size $\geq n/2$] $<$ some term that goes to zero.
   (b) The argument we showed in class, proves that there exists a c-expanding graph with out-degree $\leq d$, for some constant d. Prove that there exists such a graph with degree (in or out) bounded by d.
   (c) For $c = 1.5$, find a constant d s.t. there exists a family of c-expanding graphs with degree d and expansion c.