Homework 3
Due 2012/10/22

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1. **Tribes function:** For any k, l we define the *tribes function* $f : \{0,1\}^n \to \{-1,1\}$ on $n = kl$ variables as

   $$f(x_1, \dots, x_n) = OR(AND(x_1, \dots, x_l), AND(x_{l+1}, \dots, x_{2l}), \dots, AND(x_{(k-1)l+1}, \dots, x_{kl})).$$

   (a) Compute the influence of each of its variables.

   (b) Show that for any l, there is a way to choose k such that the tribes function is more-or-less balanced (or more precisely, that the limit of $\text{Exp}[f]$ is 0 as l goes to infinity).

   (c) Compare the maximum influence of the balanced tribes function with that of the majority function.

2. **Quasirandomness implies low correlation with juntas:**

   (a) For $f, g : \{0,1\}^n \to \mathbb{R}$ define $\text{Cov}[f,g] := \text{Exp}_x[f(x)g(x)] - \text{Exp}_x[f(x)]\,\text{Exp}_x[g(x)]$. Find an expression for $\text{Cov}[f,g]$ in term of the Fourier coefficients of f and g.

   (b) Show that for any (ε, δ)-quasirandom function $h : \{0,1\}^n \to [-1,1]$ and any r-junta $f : \{0,1\}^n \to \{-1,1\}$, $\text{Cov}[h,f] < \sqrt{\varepsilon r/(1-\delta)^r}$. Notice that this result is trivial for $r \geq \ln(1/\varepsilon)/\delta$. Hint: recall the Cauchy-Schwarz inequality $\sum a_i b_i \leq \sqrt{\sum a_i^2}\sqrt{\sum b_i^2}$.

3. **The Nisan-Szegedy bound [2]:** Let $f : \{0,1\}^n \to \mathbb{R}$ be a nonzero function of degree at most d (i.e., $\hat{f}(S) = 0$ for all S of size at least $d+1$).

   (a) Show that $\Pr[f(x) \neq 0] \geq 2^{-d}$ (this is known as the Schwartz-Zippel lemma). Hint: induction on n.

   (b) Show that if in addition f maps into $[-1,1]$ then $\mathbb{I}(f) \leq d$.

   (c) Show that if in addition f maps into $\{-1,1\}$ then f is a $d2^d$-junta.

   (d) Consider the address function $\text{Addr}_k : \{0,1\}^{k+2^k} \to \{-1,1\}$ defined by

   $$\text{Addr}_k(x_1, \dots, x_k, y_1, \dots, y_{2^k}) = (-1)^{y_x}$$

   where we think of x here as an element of $[2^k]$. Show that $\deg(\text{Addr}_k) = k+1$. Conclude that the bound in (c) must be at least $2^{d-1} + d - 1$.

4. **Unbalanced functions have a low Fourier coefficient:** Let $f : \{0,1\}^n \to \{-1,1\}$ be such that $\hat{f}(\emptyset) \notin \{-1,0,1\}$ (i.e., f is neither constant nor balanced).

   (a) Show that there must exist a nonempty S of size at most $2n/3$ such that $\hat{f}(S) \neq 0$. Hint: f^2

   (b) Optional: show that the $2n/3$ bound above is tight.

   (c) Does a similar statement hold for *balanced* functions?

5. **Bent functions:** Show an upper bound on $\|\hat{f}\|_1 := \sum_S |\hat{f}(S)|$ among all functions $f : \{0,1\}^n \to \{-1,1\}$. For infinitely many n, show a function achieving this bound.

6. **Deterministically estimating Fourier coefficients:** A set $\mathcal{A} \subseteq \{0,1\}^n$ is called ε-*biased* if for x chosen uniformly from $\mathcal{A}$ and for all nonempty $S \subseteq [n]$, $|\operatorname{Exp}_x[\chi_S(x)|| \le \varepsilon$. There is a known algorithm that on inputs ε, n, outputs an ε-biased set of size $(n/\varepsilon)^2$ in time $\operatorname{poly}(n, 1/\varepsilon)$. Use this to show how to *deterministically* estimate $\hat{f}(S)$ to within $\pm\varepsilon$ for any given S in time $\operatorname{poly}(\|\hat{f}\|_1, n, 1/\varepsilon)$ using query access to $f : \{0,1\}^n \to \mathbb{R}$. You can assume the algorithm knows $\|\hat{f}\|_1$.

7. **The Goemans-Williamson MAX-CUT** 0.87856**-approximation algorithm [1]: (no need to hand in)** The input to the algorithm is an undirected graph $G = (V, E)$ on n vertices. The first step is to solve the following optimization problem over vector variables $v_1, \ldots, v_n \in \mathbb{R}^n$: maximize $\sum_{\{i,j\} \in E} (1 - \langle v_i, v_j \rangle)/2$ subject to all vectors being unit vectors. It is known that this optimization problem can be solved efficiently (because it is a *convex optimization problem*, and in fact a *semidefinite program*). Notice that the value of the optimum is at least the number of edges in the optimal MAX-CUT. The second step in the algorithm is to take the optimal solution $v_1, \ldots, v_n$ and to construct from it a good solution to MAX-CUT (this step is known as *rounding*). This is done as follows: choose a random unit vector $w \in \mathbb{R}^n$ uniformly and partition the vertices according to the sign of $\langle w, v_i \rangle$. Notice that each edge $\{i, j\}$ is cut with probability $\frac{1}{\pi} \arccos \langle v_i, v_j \rangle$. Hence the expected size of the cut given by the algorithm is $\frac{1}{\pi} \sum_{ij} \arccos \langle v_i, v_j \rangle$. To complete the proof, notice that this is at least $\alpha \cdot \sum_{\{i,j\} \in E} (1 - \langle v_i, v_j \rangle)/2$ where $\alpha = \frac{2}{\pi} \min_{\beta \in [-1,1]} \arccos(\beta)/(1 - \beta) \approx 0.87856$.

References

[1] M. X. Goemans and D. P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. Assoc. Comput. Mach.*, 42(6):1115–1145, 1995. Preliminary version in STOC'94.

[2] N. Nisan and M. Szegedy. On the degree of Boolean functions as real polynomials. *Comput. Complexity*, 4(4):301–313, 1994. Preliminary version in STOC'92.