1. **Poincaré inequality:** Let $f : \{0,1\}^n \to \{-1,1\}$. Show that

$$\mathrm{Var}(f) = 4\Pr[f(x)=1]\Pr[f(x)=-1],$$

where $\mathrm{Var}(f)$ is as defined in Homework 1. Define the *total influence* of f to be

$$\mathbb{I}(f) = \mathop{\mathrm{Exp}}_x[\#\{i \in [n] \,:\, f(x) \neq f(x \oplus e_i)\}].$$

Show that

$$\mathbb{I}(f) \geq \mathrm{Var}(f).$$

Interpret this result as a statement about the Hamming graph $G = (V = \{0,1\}^n, E)$ with edges connecting any two vertices that differ in exactly one coordinate.

2. **Dictatorship test with perfect completeness:** Prove that there is no 3-query dictatorship test that only looks at products $f(x)f(y)f(z)$ and has perfect completeness (i.e., accepts dictatorships with probability 1) and reject functions ε-far from dictatorships with some nonzero probability.

3. **Testing resiliency:** Call a function $f : \{0,1\}^n \to \{-1,1\}$ 1-*resilient* if $\widehat{f}(S) = 0$ for all $|S| \leq 1$.

   (a) Give a combinatorial definition of 1-resiliency.

   (b) Give a $\mathrm{poly}(1/\varepsilon)$-query test that accepts 1-resilient functions with probability at least $2/3$, and rejects functions with $|\widehat{f}(S)| \geq \varepsilon$ for some $|S| \leq 1$ with probability at least $2/3$. (Notice that this is not quite the same thing as a tester for the property of being 1-resilient.) Do this by amplifying a 2-query test.

4. **Enflo's distortion lower bound on embedding ℓ_1 into ℓ_2 [2]:** The hypercube $\{0,1\}^n$ with the Hamming distance is an ℓ_1 *metric space* (because we can map $\{0,1\}^n$ to $\mathbb{R}^n$ in such a way that the Hamming distance is mapped exactly to the ℓ_1 distance). We say that the hypercube can be *embedded into ℓ_2 with distortion D* if there exists a mapping $F : \{0,1\}^n \to \mathbb{R}^m$ for some m such that for all $x, y \in \{0,1\}^n$,

$$\Delta(x,y) \leq \|F(x) - F(y)\|_2 \leq D \cdot \Delta(x,y).$$

This means that the ℓ_2 distance between $F(x)$ and $F(y)$ is the same as the Hamming distance between x and y up to a factor of D. It is easy to see that there exists an embedding with distortion $\sqrt{n}$ (think why). Here we show that this is optimal, and hence this gives an example of an ℓ_1 metric with N points whose distortion when embedded into ℓ_2 is $\sqrt{\log N}$. It was recently shown that any ℓ_1 metric with N points can be embedded into ℓ_2 with distortion $O(\sqrt{\log N}\log\log N)$ [1] (see also [3]).

   (a) Show that for any $f : \{0,1\}^n \to \mathbb{R}$,

$$\mathop{\mathrm{Exp}}_x[(f(x) - f(x \oplus (1,\ldots,1)))^2] = 4\|f^{odd}\|_2^2 \leq 4\,\mathrm{Var}[f] \leq 4\mathbb{I}(f) = \sum_{i=1}^n \mathop{\mathrm{Exp}}_x[(f(x) - f(x \oplus e_i))^2].$$

(b) Deduce that for any $F : \{0,1\}^n \to \mathbb{R}^m$,

$$\mathop{\mathrm{Exp}}_x[\|F(x) - F(x \oplus (1,\ldots,1))\|_2^2] \leq \sum_{i=1}^{n} \mathop{\mathrm{Exp}}_x[\|F(x) - F(x \oplus e_i)\|_2^2].$$

(c) Use this to conclude that the distortion of any $F : \{0,1\}^n \to \mathbb{R}^m$ must be at least $\sqrt{n}$.

5. **Compactly storing a function:** Let $f : \{0,1\}^n \to \mathbb{R}$ be some function, and assume we want to store some information about f that would allow us to compute $f(x)$ for any given $x \in \{0,1\}^n$ to within some accuracy, say, ± 0.01. Without any further restrictions on f we would have to store $\Omega(2^n)$ bits of information (even for a Boolean f).

   (a) Show how to reduce the storage to $\mathrm{poly}(n)$ for functions f with the property that for all S, $\widehat{f}(S) \geq 0$ (such functions are called *positive definite*) and moreover, $\sum_S \widehat{f}(S) = 1$. Notice that if $f = g \star g$ for some Boolean g then it satisfies these two requirements.

   (b) Extend this to functions f satisfying $\sum_S |\widehat{f}(S)| \leq \mathrm{poly}(n)$.

References

[1] S. Arora, J. R. Lee, and A. Naor. Euclidean distortion and the sparsest cut. *J. Amer. Math. Soc.*, 21(1):1–21, 2008. Preliminary version in STOC'05.

[2] P. Enflo. On the nonexistence of uniform homeomorphisms between L_p-spaces. *Ark. Mat.*, 8:103–105 (1969), 1969.

[3] J. Matoušek. Open problems on embeddings of finite metric spaces. `http://kam.mff.cuni.cz/~matousek/metrop.ps.gz`.