Homework 6

Due date: Dec 9

This homework will be graded out of 75 points even though the total points sum up to 90. Additional points on this homework will not cover up any other lost grade.

1. (30 pts) a) From the fact that column 1 + column 2 = 2(column 3), so the columns are linearly dependent, find one eigenvalue and one eigenvector of

\[ A = \begin{bmatrix} 0.2 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.3 \\ 0.4 & 0.4 & 0.4 \end{bmatrix} \] (1)

b) Find the other eigenvalues and corresponding eigenvectors to write down the eigenvalue decomposition of the matrix

\[ A = \begin{bmatrix} 0 & 10 & 0 \\ 3 & 3 & 4 \\ 1 & -1 & 0 \end{bmatrix} \]

c) If \( u_0 = (0, 10, 0)^T \), find the limit of \( A^k u_0 \) as \( k \to \infty \).

Solutions: Eigenvalues are 0, 1, -0.2. The corresponding eigenvectors are

\[ v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \] (2)

c) The solution is

\[ u_\infty = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} \] (3)

2. (30 pts) We saw in class that the solution to \( \frac{du}{dt} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} u \) (eigenvalues \( i \) and \(-i\)) goes around a circle: \( u = (\cos(t), \sin(t))^T \). Suppose we approximate \( \frac{du}{dt} \) by forward, backward and centered differences \( F, B, C \):

- (F) \( u_{n+1} - u_n = Au_n \) or \( u_{n+1} = (I + A)u_n \) (This numerical method is called Euler’s method)
- (B) \( u_{n+1} - u_n = Au_{n+1} \) or \( u_{n+1} = (I - A)^{-1}u_n \) (This numerical method is called backward Euler’s method)
- (C) \( u_{n+1} - u_n = \frac{1}{2}A(u_{n+1} + u_n) \) or \( u_{n+1} = (I - \frac{1}{2}A)^{-1}(I + \frac{1}{2}A)u_n \).

Find the eigenvalues of \( I + A, (I - A)^{-1} \) and \( (I - \frac{1}{2}A)^{-1}(I + \frac{1}{2}A) \). For which difference equation does the solution remain on a circle.

Solution: a) The eigenvalues are 1 ± i for the first case, \( \frac{1}{1\pm i} \) for the second case and 0.6 ± 0.8i in the third case. The third case leads to the solution living on the circle.

3. (30 pts) Suppose the rabbit population \( r \) and the wolf population \( w \) are governed by the following equations:

\[ \frac{dr}{dt} = 4r - 2w \] (4)

\[ \frac{dw}{dt} = r + w \] (5)

a) Is this system stable, neutrally stable or unstable?

b) If initially \( r = 300 \) and \( w = 200 \), what are the populations at time \( t \)?

c) After a long time, what is the proportion of rabbits to wolves?

Solution: The eigenvalues of the matrix are 2, 3 and the corresponding eigenvectors are

\[ v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \] (6)
b) The populations as a function of time are

\[ r = 100e^{2t} + 200e^{3t} \]  

\[ w = 100e^{2t} + 100e^{3t} \]  

(7)  

(8)

c) As \( t \to \infty \), the ratio of the rabbit to wolf population will be given by

\[
\lim_{t \to \infty} \frac{r}{w} = \lim_{t \to \infty} \frac{100e^{2t} + 200e^{3t}}{100e^{2t} + 100e^{3t}} = \lim_{t \to \infty} \frac{e^{3t}(100e^{-t} + 200)}{e^{3t}(100e^{-t} + 100)} = \lim_{t \to \infty} \frac{200}{100} = 2
\]  

(9)