Barotropic geophysical flows and two-dimensional fluid flows: Conserved Quantities

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Lecture 2: Conserved Quantities

1. Review: Some special exact solutions

2. Conserved quantities
   - Conservation of energy and enstrophy
   - Barotropic geophysical flows in a channel domain

3. Variational derivatives and an optimization principle

4. More equations for geophysical flows
Outline

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4. More equations for geophysical flows
The simplest set of equations that meaningfully describes the motion of two-dimensional geophysical flows under these circumstances is given by the:

**Barotropic Quasi-Geostrophic Equations:**

\[
\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + v \cdot \nabla q = D(\Delta) \psi + F(x, t)
\]

where

\[
q = \omega + \beta y + h(x, y), \quad \omega = \Delta \psi, \quad v = (-\partial_y \psi, \partial_x \psi)^T.
\]

- \(q\): the potential vorticity; \(\psi\): the stream function;
- \(v = \nabla^\perp \psi\): the horizontal velocity field; \(\omega = \Delta \psi\): the relative vorticity;
- \(\beta y\): the beta-plane effect from the Coriolis force;
- \(h(x)\): the bottom floor topography, given by ocean floor or mountain range;
- \(D(\Delta)\): various possible dissipation mechanisms
- \(F(x, t)\): additional external forcing.
Reduced linear system for the stream function $\psi$

Here we concentrate on finding special exact solutions with both forcing and dissipation with the stronger ansatz: we assume that $q$ and $\psi$ are linearly dependent

$$ q = \mu \psi = \Delta \psi + \beta y + h $$

The solution of the barotropic quasi-geostrophic equations under the linear dependence assumption

small and large scale of stream functions $\psi = -V_0 y + \psi'(x, y, t)$

$$ \mu \frac{\partial \psi'}{\partial t} = \mathcal{D}(\Delta) \psi' + \mathcal{F}(x, t) , $$

$$ \mu \psi' = \Delta \psi' + h(x) , \ V_0 = -\beta / \mu . $$

$$ \psi = -V_0 y + \psi' = \frac{\beta}{\mu} y + \sum_{|k| \neq 0} \hat{A}_k e^{ix \cdot k} . $$
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Conserved quantities

Conserved quantities play a decisive role both in physics and in mathematics. They are especially important in studying general properties of solutions, and they are of great importance in the non-linear stability theory and statistical theory.

Candidates for Physically Conserved Quantities:

- **Kinetic Energy**:

  \[ E = \frac{1}{2} \int |\nabla \psi|^2 + \frac{1}{2} A_R V^2 (t); \]

  - small scale
  - mean

- **Enstrophy and Potential Enstrophy**:

  \[ \mathcal{E} = \frac{1}{2} \int \omega^2, \quad \mathcal{E} = \frac{1}{2} \int q^2 ; \]

- **Large-scale Enstrophy**:

  \[ Q = (\omega + h)^2 + \beta V (t); \]

- **Generalized Enstrophy**:

  \[ Q = \int G (q). \]
Conservation of Kinetic Energy

A nature candidate of **Kinetic Energy**:

\[
E = \frac{1}{2} \int |v|^2 = \frac{1}{2} \int |\nabla \psi|^2.
\]

Recall the dynamical equation

\[
\frac{\partial q}{\partial t} + \nabla^\perp \psi \cdot \nabla q = \mathcal{D} (\Delta) \psi + \mathcal{F} (x, t)
\]

The change rate of kinetic energy in time

\[
\frac{dE}{dt} = \int \nabla \psi \cdot \nabla \psi_t = - \int \psi \frac{\partial}{\partial t} \Delta \psi = - \int \psi \frac{\partial q}{\partial t}
\]

\[
= \left\{ \int \psi \nabla^\perp \psi \cdot \nabla q \right\} + \left\{ - \int \psi \mathcal{D} (\Delta) \psi \right\} + \left\{ - \int \psi \mathcal{F} (x, t) \right\}
\]

\[
= \{1\} + \{2\} + \{3\}.
\]
Because the potential vorticity \( q, q = \Delta \psi + \beta y + h \), is not a periodic function in the presence of \( \beta y \), the actual calculation becomes subtle under periodic geometry.

Since

\[
\psi \nabla^\perp \psi \cdot \nabla q = \psi \nabla^\perp \psi \cdot \nabla (\omega + h) + \beta \psi \psi_x
= \frac{1}{2} \nabla^\perp (\psi^2) \cdot \nabla (\omega + h) + \frac{\beta}{2} (\psi^2)_x,
\]

we get

\[
\{1\} \equiv \int_{[0,2\pi] \times [0,2\pi]} \psi \nabla^\perp \psi \cdot \nabla q
= \frac{1}{2} \int_{[0,2\pi] \times [0,2\pi]} \nabla^\perp (\psi^2) \cdot \nabla (\omega + h) + \frac{\beta}{2} \int_{[0,2\pi] \times [0,2\pi]} (\psi^2)_x
= \frac{1}{2} \int_{\partial[0,2\pi] \times [0,2\pi]} (\omega + h) \nabla^\perp (\psi^2) \cdot \vec{n} + \frac{\beta}{2} \left( \int_{x=2\pi} (\psi^2) - \int_{x=0} (\psi^2) \right)
= 0,
\]

where \( \vec{n} \) denotes the unit outer-normal to the periodic domain \([0, 2\pi] \times [0, 2\pi]\). The boundary terms drop out since \( \psi, \omega, \) and \( h \) are periodic.
\{2\} = - \int \psi \mathcal{D}(\Delta) \psi = - \int \psi \sum_{j=0}^{l} d_j (-1)^j \Delta^j \psi.

Notice for \( j = 2k \) even, we have
\[
\int \psi (-1)^j \Delta^j \psi = \int \psi \Delta^{2k} \psi = \int \Delta^k \psi \Delta^k \psi = \int \Delta^{j/2} \psi \Delta^{j/2} \psi.
\]
Likewise, if \( j = 2k + 1 \) is odd, we have
\[
\int \psi (-1)^j \Delta^j \psi = - \int \psi \Delta^{2k+1} \psi = - \int \Delta^k \psi \Delta^{k+1} \psi = \int \nabla \Delta^k \psi \cdot \nabla \Delta^k \psi
= \int \nabla \Delta^{(j-1)/2} \psi \cdot \nabla \Delta^{(j-1)/2} \psi,
\]
where we utilized the classical integration by parts formula
\[
\int (\Delta F) G = - \int \nabla F \cdot \nabla G = \int F \Delta G
\]
for periodic functions \( F \) and \( G \).
Hence we have
\[
\{2\} = - \sum_{i=0. \, i \text{ even}}^{l} d_j \int |\Delta^{j/2} \psi|^2 - \sum_{i=1. \, i \text{ odd}}^{l} d_j \int |\nabla \Delta^{(j-1)/2} \psi|^2.
\]
Claim \{3\} We will assume that $\int \mathcal{F} = 0$. This means that we can rewrite $\mathcal{F}$ in the alternative form of $\mathcal{F} = \text{curl} \tilde{F}, \tilde{F} = \nabla^\perp \Phi$ or $\mathcal{F} = \Delta \Phi$. More specifically, if $\mathcal{F}$ takes the Fourier expansion

$$\mathcal{F} = \sum_{\vec{k} \neq 0} \hat{\mathcal{F}}_{\vec{k}} e^{i \vec{k} \cdot \vec{x}},$$

then $\Phi$ is given by

$$\Phi = \sum_{\vec{k} \neq 0} -\frac{\hat{\mathcal{F}}_{\vec{k}}}{|\vec{k}|^2} e^{i \vec{k} \cdot \vec{x}}$$

and we have the desired alternative forms. It then follows that

$$\{3\} = - \int \psi \mathcal{F} = - \int \psi \text{curl} \tilde{F} = \int \nabla^\perp \psi \cdot \tilde{F} = \int \vec{v} \cdot \tilde{F}$$

or

$$\{3\} = - \int \psi \mathcal{F} = - \int \psi \Delta \Phi = - \int \Delta \psi \Phi = - \int \omega \Phi.$$

Combining our estimates on the three terms and putting them back into the equation (1.74), we obtain the results on conservation of energy in the absence of the mean velocity field.
Theorem 1.1 (Kinetic energy identity) The solutions of the barotropic quasi-geostrophic equations (1.1) with periodic boundary conditions satisfy the kinetic energy identity

$$\frac{d}{dt} \frac{1}{2} \int |\vec{v}|^2 = -\mathcal{D}_E + \int \vec{v} \cdot \vec{F},$$

where

$$\mathcal{D}_E = \sum_{j=0, j: \text{even}}^l d_j \int |\Delta^{j/2} \psi|^2 + \sum_{j=1, j: \text{odd}}^l d_j \int |\nabla \Delta^{(j-1)/2} \psi|^2$$

$$\text{curl } \vec{F} = \mathcal{F}.$$ 

Corollary 1.1 In the absence of dissipation and forcing, the kinetic energy of periodic flow is conserved.
Stream Function with time dependent mean flow:

\[ \psi = -V(t)y + \psi', \]

mean \hspace{1cm} periodic

The *small-scale part* of kinetic energy (assume \( D = 0, F = 0 \))

\[
\frac{d}{dt} E_{\text{small scale}} = \frac{d}{dt} \left( \frac{1}{2} \right) |\nabla \psi'|^2 = - \int \psi' \nabla \perp \psi' \cdot \nabla q + \int V(t) \frac{\partial q}{\partial x} \psi'
\]

\[
= \frac{d}{dt} \left( \frac{1}{2} \right) \int \frac{\partial \Delta \psi'}{\partial x} \psi' + V(t) \int \frac{\partial h}{\partial x} \psi'.
\]
Large-scale and Small-scale flow interaction

Stream Function with time dependent mean flow:

\[
\psi = -V(t)y + \psi' .
\]

The small-scale part of kinetic energy (assume \( D = 0, F = 0 \))

\[
\frac{d}{dt} E_{\text{small scale}} = \frac{d}{dt} \frac{1}{2} \int |\nabla \psi'|^2 = - \int \psi' \nabla \psi' \cdot \nabla q + \int V(t) \frac{\partial q}{\partial x} \psi' \\
= V(t) \int \frac{\partial \Delta \psi'}{\partial x} \psi' + V(t) \int \frac{\partial h}{\partial x} \psi'.
\]

Notice that

\[
\int \frac{\partial \Delta \psi'}{\partial x} \psi' = - \int \frac{\partial}{\partial x} (\nabla \psi') \cdot \nabla \psi' = - \int \frac{\partial}{\partial x} \left( \frac{1}{2} |\nabla \psi'|^2 \right) = 0.
\]
Large-scale and Small-scale flow interaction

Stream Function with time dependent mean flow:

\[ \psi = -V(t)y + \psi'. \]

The small-scale part of kinetic energy (assume \(D = 0, \mathcal{F} = 0\))

\[
\frac{d}{dt} E_{\text{small scale}} = \frac{d}{dt} \frac{1}{2} \int |\nabla \psi'|^2 = -\int \psi' \nabla^\perp \psi' \cdot \nabla q + \int V(t) \frac{\partial q}{\partial x} \psi'
\]

\[
= V(t) \int \frac{\partial \Delta \psi'}{\partial x} \psi' + V(t) \int \frac{\partial h}{\partial x} \psi'.
\]

Therefore we deduce

\[
\frac{d}{dt} E_{\text{small scale}} = V(t) \int \frac{\partial h}{\partial x} \psi' \rightarrow \text{topographic stress}
\]
Since there is no dissipation and forcing, it is natural to postulate the assumption that the total energy is conserved

\[ 0 \equiv \frac{d}{dt} E_{\text{total}} = \frac{d}{dt} A_R \frac{1}{2} V^2(t) + \frac{d}{dt} E_{\text{small scale}} \]

\[ = A_R V(t) \frac{d}{dt} V(t) + V(t) \int \frac{\partial h}{\partial x} \psi'. \]
Since there is no dissipation and forcing, it is natural to postulate the assumption that the total energy is conserved

\[ 0 \equiv \frac{d}{dt} E_{\text{total}} = \frac{d}{dt} A_R \frac{1}{2} V^2 (t) + \frac{d}{dt} E_{\text{small scale}} = A_R V (t) \frac{d}{dt} V (t) + V (t) \int \frac{\partial h}{\partial x} \psi'. \]

This implies in the case \( V \neq 0 \)

\[ \frac{d}{dt} V (t) = -\frac{\partial h}{\partial x} \psi' \]

with \( \Omega f = \frac{1}{A_R} \int f. \)
Extension for large-scale and small-scale flow interaction via topographic stress

\[ A) \quad q = \Delta \psi' + h + \beta y \]

\[ B) \quad \psi = -V(t) y + \psi' \]

\[ C) \quad \frac{\partial q}{\partial t} + J(\psi, q) = 0 \]

\[ D) \quad \frac{d}{dt} V(t) = -\frac{\partial h}{\partial x} \psi' \]

Steady state solution \( \mu \psi' = \Delta \psi' + h \) remains true

\[
\int \frac{\partial h}{\partial x} \psi' = -\int h \frac{\partial \psi'}{\partial x} = \int \Delta \psi' \frac{\partial \psi'}{\partial x} - \mu \int \psi' \frac{\partial \psi'}{\partial x}
\]

\[
= \frac{1}{2} \int \frac{\partial}{\partial x} \left( |\nabla \psi'|^2 - \mu (\psi')^2 \right) = 0.
\]
Conservation of Large-Scale Enstrophy

**Large-Scale Enstrophy:**

\[
Q = \beta V(t) + \frac{1}{2} (\omega + h)^2.
\]

\[
\frac{d}{dt} \frac{1}{2} \int (\omega + h)^2 = \int (\omega + h) \frac{\partial \omega}{\partial t}
\]

\[
= -\int (\omega + h) \nabla \psi' \cdot \nabla (\omega + h + \beta y) - V(t) \int (\omega + h) \frac{\partial}{\partial x} (\omega + h + \beta y)
\]

\[
= -\frac{1}{2} \int \nabla \psi' \cdot \nabla (\omega + h)^2 - \beta \int (\omega + h) \frac{\partial \psi'}{\partial x} - \frac{V(t)}{2} \int \frac{\partial}{\partial x} (\omega + h)^2
\]

\[
= -\beta \int \Delta \psi' \frac{\partial \psi'}{\partial x} - \beta \int h \psi' = \frac{\beta}{2} \int \frac{\partial}{\partial x} |\nabla \psi'|^2 + \beta \int \frac{\partial h}{\partial x} \psi'
\]

\[
= -\beta A_R \frac{dV(t)}{dt},
\]

since \(\psi', \omega, h\) are periodic. Thus \(Q\) is also conserved.
Conservation of Generalized Enstrophy

\[ \int_{[0,2\pi] \times [0,2\pi]} G(q) \, d \theta \, d\phi. \]

\[
\frac{d}{dt} \int_{[0,2\pi] \times [0,2\pi]} G(q) \, d\theta \, d\phi = \int_{[0,2\pi] \times [0,2\pi]} G'(q) \frac{\partial q}{\partial t} \, d\theta \, d\phi
\]

\[
= -\int_{[0,2\pi] \times [0,2\pi]} (\tilde{v} \cdot \nabla) G'(q) \, d\theta \, d\phi
\]

\[
= -\int_{[0,2\pi] \times [0,2\pi]} (\tilde{v} \cdot \nabla) G(q) \, d\theta \, d\phi
\]

\[
= -\int_{\partial [0,2\pi] \times [0,2\pi]} (\tilde{v} \cdot \tilde{n}) G(q) \, d\theta \, d\phi
\]

\[
= -\int_{y=2\pi} \psi_x G(\Delta \psi + h + \beta y) + \int_{y=0} \psi_x G(\Delta \psi + h) \, d\theta \, d\phi
\]

\[
= -\int_{0}^{2\pi} \psi_x(x, 0, t) (G(\Delta \psi(x, 0, t) + h(x, 0) + 2\pi\beta)
\]

\[-G(\Delta \psi(x, 0, t) + h(x, 0)) \, dx).\]
Conservation of Generalized Enstrophy

\[
\frac{d}{dt} \int_{[0,2\pi] \times [0,2\pi]} G(q) = - \int_0^{2\pi} \psi_x(x,0,t) [G(\Delta \psi(x,0,t) + h(x,0) + 2\pi \beta) - G(\Delta \psi(x,0,t) + h(x,0))] \, dx.
\]

The generalized enstrophy is conserved if and only if one of the following is true:

- \( \beta = 0 \), i.e. there is no beta-effect;
- \( G \) is periodic with period \( 2\pi \beta \);
- \( \psi_x(x,0,t) \equiv 0 \), i.e. there exists special symmetry in the flow.
Summary of conserved quantities: periodic geometry

**Sub-case A:** $\beta \neq 0, \ h \equiv 0, \ V(t) = 0$.

(i) Energy: $E \equiv \frac{1}{2} \int |\nabla \psi|^2$.

(ii) Enstrophy: $\frac{1}{2} \int \omega^2$.

(iii) Non-robust conserved quantities: $\int_0^{2\pi} \int_0^{2\pi} G(q) \, dx \, dy$, if $G(q)$ is periodic with period $\beta 2\pi$.

**Sub-case B:** $\beta \neq 0, \ h \neq 0, \ V(t) \equiv 0$.

We neglect the large mean equation in this case, since we have an independent dynamics for the small scale.

(i) **Single** robust conserved quantity energy: $E \equiv \frac{1}{2} \int |\nabla \psi|^2$.

(ii) Neither enstrophy $\frac{1}{2} \int \omega^2$ nor potential enstrophy $\frac{1}{2} \int q^2$ are conserved.

(iii) Non-robust conserved quantities: $\int_0^{2\pi} \int_0^{2\pi} G(q) \, dx \, dy$, provided $G(q)$ is periodic with period $\beta 2\pi$.

**Sub-case C:** $\beta \neq 0, \ h \neq 0, \ V(t) \neq 0$.

(i) Total mean energy: $E \equiv \frac{1}{2} V^2 + \frac{1}{2} \int |\nabla \psi'|^2$.

(ii) Large-scale enstrophy: $\beta V(t) + \frac{1}{2} \int (\omega + h)^2$.

(iii) Non-robust conserved quantities: $\int_0^{2\pi} \int_0^{2\pi} G(q) \, dx \, dy$, provided $G(q)$ is periodic with period $\beta 2\pi$. 

It is tempting to conclude that the extrema (maximum and minimum) of the potential vorticity are conserved, since the potential vorticity is advected along the stream lines. However this is not true, since the potential vorticity is not periodic if $\beta \neq 0$, and the stream lines may escape the periodic domain. It is easy to cook up counter-examples where the extrema are not conserved. Another way to understand this is to view the problem as flow on the whole plane with periodic structure. Then the potential vorticity is an unbounded quantity on the whole plane. We will see later in this chapter that it will be very different in the case of channel geometry.
Channel domain geometry

Domain is given by the regime:

\[ 0 < x < 2\pi, \quad 0 < y < \pi. \]

Boundary conditions:

\[ \vec{v} (x + 2\pi, y, t) = \vec{v} (x, y, t), \]
\[ v_2 (x, \pi, t) = v_2 (x, 0, t) = 0. \]
Channel domain geometry

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Boundary conditions:

\[ \vec{v} (x + 2\pi, y, t) = \vec{v} (x, y, t), \]
\[ v_2 (x, \pi, t) = v_2 (x, 0, t) = 0. \]

The solid wall boundary condition implies that the stream function satisfies:

\[ \psi (x, \pi, t) = A (t), \quad \psi (x, 0, t) = B (t) = 0. \]

The most general stream function satisfying the boundary condition has the form

\[ \psi (x, y) = -V y + \psi' (x, y), \quad \psi' |_{y=0,\psi=0}, \]

where \( V = -A/\pi. \)
Connection with the periodic geometry

We extend $\psi'$ oddly in the $y$ direction to the box $[0, 2\pi] \times [-\pi, \pi]$, i.e.

$$\tilde{\psi}'(x, y) = \begin{cases} 
\psi'(x, y), & \text{for } y > 0, \\
-\psi'(x, -y), & \text{for } y \leq 0.
\end{cases}$$

- Both $\tilde{\psi}'$ and the small-scale velocity $\nabla \cdot \tilde{\psi}'$ are periodic functions in $x$ and $y$ thanks to the odd extension.
- The Fourier representation of $\tilde{\psi}'$ cannot contain even terms in $y$

$$\psi'(x, y) = \sum_{k \geq 1} \sum_{j} (a_{jk} \cos(jx) + b_{jk} \sin(jx)) \sin(ky).$$
Barotropic quasi-geostrophic equations in a channel with topographic stress

\[
q = \Delta \psi' + h + \beta y, \quad \psi = -V(t)y + \psi',
\]

\[
\frac{\partial q}{\partial t} + J(\psi, q) = 0, \quad \frac{d}{dt} V(t) = -\int \frac{\partial h}{\partial x} \psi'.
\] (1.105)

We note here that the area for the domain is \(2\pi^2\) and hence \(2\pi^2\) is now the one utilized in the normalized integral in the dynamic equation for the large mean \(V\).

Conserved quantities

(i) Total mean energy

\[
E = \frac{1}{2} V^2(t) + \frac{1}{2} \int |\nabla \psi'|^2.
\] (1.106)

(ii) Large-scale enstrophy

\[
Q = \beta V(t) + \frac{1}{2} \int (\omega + h)^2.
\] (1.107)

(iii) Generalized enstrophies

\[
\int G(q)
\]

for an arbitrary function \(G(q)\), where \(q = \Delta \psi' + h + \beta y\).
Conservation of extrema of the potential vorticity

Mathematically we let \( m \) be the maximum of the initial potential vorticity in the channel

\[
m = \max q(\cdot, 0)
\]  

(1.109)

and we define

\[
G(q) = \begin{cases} 
(q - m)^2, & \text{if } q \geq m \\
0, & \text{if } q < m.
\end{cases}
\]

(1.110)

Then \( G \) is differentiable and hence \( \mathcal{G} = \int G(q) \) is conserved. Notice that \( \mathcal{G}(0) = 0 \) since \( m \) is the maximum of the initial potential vorticity. Thus \( \mathcal{G}(t) = 0 \) for all \( t \), which further implies

\[
q(\cdot, t) \leq m
\]

(1.111)

The conservation of the minimum can be verified the same way. Note that this argument also shows that we have conservation of the extrema for general periodic flows with \( \beta = 0 \).
For any $\epsilon > 0$

$$G_\epsilon(q) = \begin{cases} (q - m + \epsilon)^2, & \text{if } q \geq m - \epsilon \\ 0, & \text{if } q < m - \epsilon. \end{cases}$$ (1.112)

Then $G_\epsilon$ is differentiable and hence $G_\epsilon = \int G_\epsilon(q)$ is conserved. Notice that $G_\epsilon(0) > 0$ since $m$ is the maximum of the initial potential vorticity and $\epsilon > 0$. Thus $G_\epsilon(t) > 0$ for all $t$, which further implies

$$\max q(\cdot, t) \geq m - \epsilon$$ (1.113)

for all $t$ and all $\epsilon > 0$. Hence

$$\max q(\cdot, t) \geq m,$$ (1.114)

which leads to

$$\max q(\cdot, t) = m = \max q(\cdot, 0).$$
The impulse and conserved quantities

The *impulse*, $I$, defined by

$$I = y\omega, \quad \text{where } \omega = \Delta \psi'.$$

For $\beta \neq 0$, the standard choice of the conserved potential enstrophy, $G(q) = \frac{1}{2} q^2$, yields the conserved quantity

$$\frac{1}{2} \int (\omega + h)^2 + \beta I,$$

since $q = \omega + h + \beta y$. The difference of this conserved quantity and the conserved large-scale enstrophy $Q$ yields the fact that for $\beta \neq 0$

$$V(t) - I(t) \quad \text{is conserved in time.}$$

The impulse represents the mean $x$-momentum (horizontal flux rate) induced by the relative vorticity in the fluid.
The impulse and conserved quantities

The impulse, I, defined by

\[ I = y\omega, \quad \text{where } \omega = \Delta\psi'. \]

For \( \beta \neq 0 \), the standard choice of the conserved potential enstrophy, \( G(q) = \frac{1}{2}q^2 \), yields the conserved quantity

\[ \frac{1}{2} \int (\omega + h)^2 + \beta I, \]

since \( q = \omega + h + \beta y \). The difference of this conserved quantity and the conserved large-scale enstrophy \( Q \) yields the fact that for \( \beta \neq 0 \)

\[ V(t) - I(t) \quad \text{is conserved in time.} \] (1.118)

In particular, for a zonal topography,

\[ I, \quad \text{is conserved in time, if } h = h(y). \]
Conservation of circulation

The \textit{circulation}, \( \Gamma \), defined by

\[
\Gamma = \int \omega, \quad \text{where} \quad \omega = \Delta \psi'.
\]

This measures the total vorticity (or the global rotation of the fluid) in the domain. It is called circulation since

\[
\Gamma = \int \left( -\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) = \int_{\partial \Omega} \left( -n_2 v_1 + v_2 n_1 \right) ds = \int_{\partial \Omega} \vec{v} \cdot \vec{\tau} ds
\]

\[
= -\Gamma_{\text{top}} + \Gamma_{\text{bottom}},
\]

where \( \vec{\tau} \) is the unit counter-clockwise tangent vector at the boundary of the channel and \( \Gamma_{\text{top}} = \int_{y=\pi} v_1 \, dx \), \( \Gamma_{\text{bottom}} = \int_{y=0} v_1 \, dx \).
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Variational Derivatives

For an ordinary differential function $F$ from $\mathbb{R}^N$ to $\mathbb{R}$, the directional derivatives satisfy

$$\lim_{\epsilon \to 0} \frac{F(x + \epsilon y) - F(x)}{\epsilon} \overset{\text{def.}}{=} \nabla F(x) \cdot y.$$ 

Given a functional, $\mathcal{F}(u)$, from a Hilbert space $H$ to $\mathbb{R}$, the variational derivative of $\mathcal{F}$, denoted as $\frac{\delta \mathcal{F}}{\delta u}$, is the function satisfying

$$\lim_{\epsilon \to 0} \frac{\mathcal{F}(u + \epsilon \delta u) - \mathcal{F}(u)}{\epsilon} \overset{\text{def.}}{=} \left\langle \frac{\delta \mathcal{F}}{\delta u}, \delta u \right\rangle,$$

for any $\delta u$. The inner product space will be ordinary integrable functions on a two-dimensional domain

$$\langle f, g \rangle = (f, g)_0 = fg.$$
Some important variational derivatives

There are some elementary variational derivatives which play an important role.

1. Kinetic Energy

\[ E(\omega) = \frac{1}{2} |v|^2 = \frac{1}{2} |\nabla \psi|^2 = -\frac{1}{2} \psi \omega. \]

\[
\frac{E(\omega + \varepsilon \delta \omega) - E(\omega)}{\varepsilon} = -\frac{1}{2} \int (\psi \delta \omega + \delta \psi \omega) - \varepsilon \frac{1}{2} \int \delta \psi \delta \omega
\]

(1.135)

so that

\[
\lim_{\varepsilon \to 0} \frac{E(\omega + \varepsilon \delta \omega) - E(\omega)}{\varepsilon} = -\frac{1}{2} \int (\psi \delta \omega + \delta \psi \omega) = -\frac{1}{2} \int (\psi \delta \omega + \delta \psi \Delta \psi)
\]

\[
= -\int \psi \delta \omega = (-\psi, \delta \omega)_0
\]

\[
\frac{\delta E}{\delta \omega} = -\psi, \quad \Delta \psi = \omega
\]
2. Generalized Enstrophy

\[ \mathcal{G}(\omega) = \int G(\omega) \]  

(1.137)

for a differentiable function \( G \). Here it is easy to see that

\[ \lim_{\varepsilon \to 0} \frac{\mathcal{G}(\omega + \varepsilon \delta \omega) - \mathcal{G}(\omega)}{\varepsilon} = \int G'(\omega) \delta \omega \]  

(1.138)

so that

\[ \frac{\delta \mathcal{G}}{\delta \omega} = G'(\omega). \]  

(1.139)
An optimization principle for elementary geophysical solutions

We consider the averaged enstrophy

$$\mathcal{E}(\omega) = \frac{1}{2} (\omega + h)^2,$$

and ask the question:

*Which functions, $\omega$, minimize the enstrophy for a fixed value of the energy, $E(\omega) = E_0$?*
An optimization principle for elementary geophysical solutions

We consider the averaged enstrophy

\[ \mathcal{E}(\omega) = \frac{1}{2} (\omega + h)^2, \]

and ask the question:

*Which functions, \( \omega \), minimize the enstrophy for a fixed value of the energy, \( \mathcal{E}(\omega) = E_0 \)?*

According to the *Lagrange multiplier principle*, the function \( \omega \) should satisfy the *Euler-Lagrange equation*

\[ \frac{\delta \mathcal{E}}{\delta \omega} = \mu \frac{\delta \mathcal{E}}{\delta \omega}, \]

where \( \mu \) is a Lagrangian multiplier.
An optimization principle for elementary geophysical solutions

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where \( \mu \) is a Lagrangian multiplier.

The Euler-Lagrange equation becomes

\[ \Delta \psi + h = \mu \psi. \]

*Note that, as in ordinary calculus, not every solution is necessarily a minimize of the enstrophy, and some of the solutions are saddle points.*
Outline

1 Review: Some special exact solutions

2 Conserved quantities
   - Conservation of energy and enstrophy
   - Barotropic geophysical flows in a channel domain

3 Variational derivatives and an optimization principle

4 More equations for geophysical flows
The models

F-plane equations

\[
\frac{\partial q}{\partial t} + J(\psi, q) = D(\Delta) \psi + \mathcal{F},
\]

\[q = \Delta \psi - F^2 \psi + h + \beta y,
\]

where \( F = L / L_R \) with \( L \) the *characteristic horizontal length*, \( L_R = \sqrt{gH_0/f_0} \) the *Rossby deformation radius*.

Continuously stratified quasi-geostrophic equations

\[
\frac{\partial}{\partial t} \left( \Delta \psi + F^2 \frac{\partial^2 \psi}{\partial z^2} \right) + J\left( \psi, \Delta \psi + F^2 \frac{\partial^2 \psi}{\partial z^2} \right) + \beta \frac{\partial \psi}{\partial x} = D\psi + \mathcal{F},
\]

where \( \psi(x, y, z, t) \) is defined in the domain \([0, 2\pi] \times [0, 2\pi] \times [0, 2\pi\Theta] \).
The two-layer model

Two-layer model

\[
\frac{\partial}{\partial t} (\Delta \psi_1 - F_1 (\psi_1 - \psi_2)) + J (\psi_1, \Delta \psi_1 - F_1 (\psi_1 - \psi_2)) + \beta \frac{\partial \psi_1}{\partial x} = \mathcal{F},
\]

\[
\frac{\partial}{\partial t} (\Delta \psi_2 + F_2 (\psi_1 - \psi_2)) + J (\psi_2, \Delta \psi_2 + F_2 (\psi_1 - \psi_2)) + \beta \frac{\partial \psi_2}{\partial x} = -r \Delta \psi_2 + \mathcal{F}.
\]

snapshot of barotropic vorticity, \(q_\psi\)
Relationships between various models

\[ \psi(x, y, z, t) = \psi_b(x, y, t) + \psi_t(x, y, t) \sqrt{2} \sin z. \]  

(1.151)

Substitute this into the continuously stratified model (1.146) and notice for the Jacobian we have

\[
J \left( \psi, \Delta \psi + F^2 \frac{\partial^2 \psi}{\partial z^2} \right) = J(\psi_b + \psi_t \sqrt{2} \sin z, \Delta \psi_b + (\Delta \psi_t - F^2 \psi_t) \sqrt{2} \sin z)
\]

\[
= J(\psi_b, \Delta \psi_b) + \sqrt{2} \sin z (J(\psi_t, \Delta \psi_t) \\
+ J(\psi_b, \Delta \psi_t - F^2 \psi_t)) + 2 \sin^2 z J(\psi_t, \Delta \psi_t - F^2 \psi_t).
\]

The projection of the right-hand side on to the first two vertical Fourier bases \{1, \sqrt{2} \sin z\} yields, following components for each modes

\[
\frac{\partial}{\partial t} \Delta \psi_b + J(\psi_b, \Delta \psi_b) + J(\psi_t, \Delta \psi_t - F^2 \psi_t) + \beta \frac{\partial}{\partial x} \psi_b = 0, \quad (1.152)
\]

\[
\frac{\partial}{\partial t} (\Delta \psi_t - F^2 \psi_t) + J(\psi_t, \Delta \psi_b) + J(\psi_b, \Delta \psi_t - F^2 \psi_t) + \beta \frac{\partial}{\partial x} \psi_t = 0. \quad (1.153)
\]
The well-known two-layer model ten follows:

\[
\begin{align*}
\frac{\partial}{\partial t}(\Delta (\psi_b + \psi_t) - F^2 \psi_t) + J(\psi_b + \psi_t, \Delta (\psi_b + \psi_t) - F^2 \psi_t) \\
+ \beta \frac{\partial}{\partial x} (\psi_b + \psi_t) &= 0, \\
\frac{\partial}{\partial t}(\Delta (\psi_b - \psi_t) + F^2 \psi_t) + J(\psi_b - \psi_t, \Delta (\psi_b - \psi_t) + F^2 \psi_t) \\
+ \beta \frac{\partial}{\partial x} (\psi_b - \psi_t) &= 0.
\end{align*}
\]

(1.154)
Questions & Discussions