Stochastic Models for Turbulence
Majda-Harlim Chapter 5
Outline

1. Motivation and a Test Model for Turbulence
2. Turbulent signals with Forcing and Dissipation
3. Statistics of Turbulent Solutions in Physical Space
4. Turbulent Rossby Waves
Motivation

- Turbulent flows are highly irregular and need to be analyzed in a statistical sense. This lends itself to the idea of modeling turbulence as a stochastic process.
- Coarse grained models are unable to resolve small scale turbulence.
- Large physical systems often have small scale turbulent processes which are governed by unknown dynamics.

We choose to parametrize resolved and unresolved turbulence with spatially correlated white noise forcing.
A Stochastic Test Model for Turbulent Signals

Consider the initial value problem for the following scalar stochastic PDE:

\[
\frac{\partial u(x,t)}{\partial t} = P\left(\frac{\partial}{\partial x}\right)u(x, t) - \gamma\left(\frac{\partial}{\partial x}\right)u(x, t) + \bar{F}(x, t) + \sigma(x)\dot{W}(t)
\]

\[u(x, 0) = u_0(x)\]

where,

- \(P\left(\frac{\partial}{\partial x}\right)\) is an operator constructed from odd derivatives
- \(\gamma\left(\frac{\partial}{\partial x}\right)\) is an operator constructed from even derivatives
- \(\bar{F}(x, t)\) is a deterministic forcing
- \(\sigma(x)\dot{W}(t)\) is spatially correlated white-noise forcing
- \(u_0(x) \sim N(\bar{x}, \sigma^2)\)
Test Model Operators

The operators satisfy

\[ P \left( \frac{\partial}{\partial x} \right) e^{ikx} = \tilde{p}(ik) e^{ikx} \]
\[ \gamma \left( \frac{\partial}{\partial x} \right) e^{ikx} = \gamma(ik) e^{ikx} \]

Assume \( \tilde{p}(ik) \) is wave-like:

\[ \tilde{p}(ik) = i\omega_k \]

where \( -\omega_k \) is the dispersion relation.
\( \gamma(ik) \) is the damping operator and satisfies

\[ \gamma(ik) > 0 \ \forall \ k \neq 0 \]
The Stochastically Forced Dissipative Advection Equation

Taking \( P\left( \frac{\partial}{\partial x} \right) = -c \frac{\partial}{\partial x} \) and \( \gamma\left( \frac{\partial}{\partial x} \right) = -d + \mu \frac{\partial^2}{\partial x^2} \), we have,

\[
\frac{\partial u(x, t)}{\partial t} = -c \frac{\partial u(x, t)}{\partial x} - du(x, t) + \mu \frac{\partial^2 u(x, t)}{\partial x^2} + \bar{F}(x, t) + \sigma(x) \dot{W}(t)
\]

- \( \tilde{p}(ik) = i\omega_k = -i ck \)
- \( \gamma(ik) = d + \mu k^2 \)

We seek a Fourier series solution:

\[
u(x, t) = \sum_{k=-\infty}^{\infty} \hat{u}_k(t) e^{ikx}
\]

where \( \hat{u}_{-k} = \hat{u}_k^* \)
Fourier Series Solution

\[
\frac{\partial u(x,t)}{\partial t} = P \left( \frac{\partial}{\partial x} \right) u(x,t) - \gamma \left( \frac{\partial}{\partial x} \right) u(x,t) + \bar{F}(x,t) + \sigma(x) \dot{W}(t)
\]

Each \( \hat{u}_k(t) \) satisfies the SDE:

\[
d\hat{u}_k(t) = [\tilde{p}(ik) - \gamma(ik)] \hat{u}_k(t) dt + \hat{F}_k(t) dt + \sigma_k dW_k(t)
\]

Multiplying by the integrating factor \( e^{(\gamma(ik) - \tilde{p}(ik))t} \), we have,

\[
d \left( e^{(\gamma(ik) - \tilde{p}(ik))t} \hat{u}_k(t) \right) = e^{(\gamma(ik) - \tilde{p}(ik))t} \left( \hat{F}_k(t) dt + \sigma_k dW_k(t) \right)
\]

\[
e^{(\gamma(ik) - \tilde{p}(ik))t} \hat{u}_k(t) - \hat{u}_k(0) = \\
\int_0^t e^{(\gamma(ik) - \tilde{p}(ik))s} \hat{F}_k(s) ds + \sigma_k \int_0^t e^{(\gamma(ik) - \tilde{p}(ik))s} dW_k(s)
\]

Thus,

\[
\hat{u}_k(t) = \hat{u}_k(0) e^{(\tilde{p}(ik) - \gamma(ik))t} + \int_0^t e^{(\gamma(ik) - \tilde{p}(ik))(s-t)} \hat{F}_k(s) ds + \\
\sigma_k \int_0^t e^{(\gamma(ik) - \tilde{p}(ik))(s-t)} dW_k(s)
\]
Large Time Behaviour

Taking $\bar{F}(x, t) = 0$, we have

$$\hat{u}_k(t) = \hat{u}_k(0)e^{(\bar{p}(ik)-\gamma(ik))t} + \sigma_k \int_0^t e^{(\gamma(ik)-\bar{p}(ik))(s-t)}dW_k(s)$$

Note: $W_k = \frac{W_1+iW_2}{\sqrt{2}}$

$$E[\hat{u}_k(t)] = \hat{u}_k(0)e^{(\bar{p}(ik)-\gamma(ik))t} \to 0$$

$$E[\hat{u}_k(t)\hat{u}_k(t)^*] = \hat{u}_k(0)\hat{u}_k(0)^*e^{-2\gamma(ik)t} + \frac{\sigma_k\sigma_k^*}{2\gamma(ik)}(1 - e^{-2\gamma(ik)t}) \to \frac{\sigma_k^2}{2\gamma(ik)}$$

We define the energy spectrum,

$$E_k = \frac{\sigma_k^2}{2\gamma(ik)} \quad 1 \leq k < \infty$$
Autocorrelation Function

Let $\lambda(ik) = \gamma(ik) - \tilde{p}(ik)$

Note that: $\lambda^*(ik) = \gamma(ik) + \tilde{p}(ik)$

\[ R_k(t, t + \tau) = \mathbb{E}[(\hat{u}_k(t) - \tilde{u}_k)(\hat{u}_k(t + \tau) - \tilde{u}_k)] \]
\[ = \mathbb{E}[(\sigma_k \int_0^t e^{\lambda(ik)(s-t)} dW_k(s)) (\sigma_k \int_0^{t+\tau} e^{\lambda(ik)(s'-t-\tau)} dW_k(s'))^*] \]
\[ = \sigma_k^2 e^{-\gamma(ik)(2t+\tau) - \tilde{p}(ik)\tau} \int_0^t \int_0^{t+\tau} e^{\lambda(ik)s + \lambda^*(ik)s'} \frac{1}{2} \mathbb{E}[dW_1(s)dW_1(s') + dW_2(s)dW_2(s')] \]
\[ = \sigma_k^2 e^{-\gamma(ik)(2t+\tau) - \tilde{p}(ik)\tau} \int_0^t \int_0^{t+\tau} e^{\lambda(ik)s + \lambda^*(ik)s'} \delta(s - s') ds ds' \]
\[ = \sigma_k^2 e^{-\gamma(ik)(2t+\tau) - \tilde{p}(ik)\tau} \int_0^t \int_0^{t+\tau} e^{\lambda(ik)s + \lambda^*(ik)s} ds \]
\[ = \sigma_k^2 e^{-\gamma(ik)(2t+\tau) - \tilde{p}(ik)\tau} \frac{1}{2\gamma(ik)}(e^{2\gamma(ik)t} - 1) \]
\[ = \frac{\sigma_k^2}{2\gamma(ik)} e^{-\gamma(ik)(2t+\tau) - \tilde{p}(ik)\tau} (e^{2\gamma(ik)t} - 1) \]
\[ = \frac{\sigma_k^2}{2\gamma(ik)} e^{-\gamma(ik)\tau - \tilde{p}(ik)\tau} (1 - e^{-2\gamma(ik)t}) \]
Finally, we have,

$$R(t, t + \tau) = \frac{\sigma_k^2}{2\gamma(ik)} e^{-\gamma(ik)\tau - \bar{p}(ik)\tau} (1 - e^{-2\gamma(ik)t})$$

In the large $t$ limit,

$$R(t, t + \tau) = \frac{\sigma_k^2}{2\gamma(ik)} e^{-\gamma(ik)\tau - \bar{p}(ik)\tau}$$

Thus,

$$\text{Real}(R(t, t + \tau)) = \frac{\sigma_k^2}{2\gamma(ik)} e^{-\gamma(ik)\tau} \cos(\omega_k \tau) = E_k e^{-\gamma(ik)\tau} \cos(\omega_k \tau)$$

Decorrelation Time: $\frac{1}{\gamma(ik)}$
Calibrating the Noise Level

From observations, we can roughly determine the energy spectrum and decorrelation time at each wavenumber.

Recall that \( E_k = \frac{\sigma_k^2}{2\gamma(ik)} \). Thus, we can produce an estimate of the noise level using

\[
\sigma_k = \sqrt{2\gamma(ik)E_k}
\]

A typical turbulent energy spectra has the power law form:

\[ E_k = E_0|k|^{-\beta} \]

For example, if \( \gamma(ik) = d + \mu k^2 \),

\[
\sigma_k = E_0^{1/2}|k|^{-\beta/2}(d + \mu k^2)^{1/2}
\]

- \( \beta > 2 \Rightarrow \) decreasing noise at small spatial scales
- \( \beta < 2 \Rightarrow \) increasing noise at small spatial scales
Damped Forced Solutions

Consider a forcing of the following form:

\[ \hat{F}_k(t) = \begin{cases} \ Ae^{i\omega_0(k)t} & k \leq M \\ 0 & k > M \end{cases} \]

The mean dynamics satisfy:

\[ \bar{\hat{u}}_k(t) = \bar{\hat{u}}_k(0)e^{-\lambda(i\kappa)t} + \int_0^t e^{\lambda(i\kappa)(s-t)}\hat{F}_k(s)ds \]

\[ = \begin{cases} \bar{\hat{u}}_k(0)e^{-\lambda(i\kappa)t} + \frac{Ae^{i\omega_0(k)t}}{\gamma(i\kappa)+i(\omega_0(k)-\omega_k)}(1 - e^{(-\lambda(i\kappa)-\omega_0(k))t}) & k \leq M \\ \bar{\hat{u}}_k(0)e^{-\lambda(i\kappa)t} & k > M \end{cases} \]
Test Problem with Resonant Forcing

We choose a resonant forcing: \( \omega_0(k) = \omega_k \ \forall \ k \) such that \( |k| \leq M \)

Numerical integrations were performed on:

\[
\frac{\partial u(x,t)}{\partial t} = -c \frac{\partial u(x,t)}{\partial x} - du(x, t) + \mu \frac{\partial^2 u(x,t)}{\partial x^2} + \bar{F}(x, t) + \sigma(x) \dot{W}(t)
\]

with parameters:

- \( c = 1 \)
- \( d = 0 \)
- \( \mu = 0.01 \)
- \( A = 0.1 \)
- \( M = 20 \)
- \( k_{\text{max}} = 61 \)
- \( E_k = 1, \ E_k = k^{-5/3} \)
- \( \sigma_k = \sqrt{0.02k}, \ \sigma_k = \sqrt{0.02k^{1/6}} \)
Test Problem with Resonant Forcing

$E_k=1$ at time=1

$E_k=k^{-5/3}$ at time=1

$E_k=1$ at time=5

$E_k=k^{-5/3}$ at time=5

$E_k=1$ at time=10

$E_k=k^{-5/3}$ at time=10
Statistics of Turbulent solutions in physical space

We have the solution

\[ u(x, t) = \sum_{|k| \leq N} \hat{u}_k(t) e_k(x). \]

Statistical behavior - Mean

\[ \langle u(x, t) \rangle = \langle \sum_{|k| \leq N} \hat{u}_k(t) e_k(x) \rangle \]
\[ = \sum_{|k| \leq N} \langle \hat{u}_k(t) \rangle e_k(x) \]
\[ = \sum_{|k| \leq N} \left( \hat{u}_k(0)e^{ip(ik)t} + \int_0^t \hat{F}_k(s)e^{ip(ik)(t-s)} ds \right) e_k(x), \]

where \( p(ik) = \bar{p}(ik) - \gamma(ik). \)

\[ \hat{u}_k(t) - \langle \hat{u}_k(t) \rangle = \bar{\sigma}_k \int_0^t e^{p(ik)(t-s)} dW_k(t), \]

where \( dW_k(t) = 2^{-1/2}(dW_{k,1}(t) + idW_{k,2}(t)) \) and each component \( W_{k,i}(t) \) is a Wiener process.
temporal correlation function

\[ R(x, x, t, t') = \sum_{|k| \leq N} \frac{\tilde{\sigma}_k^2}{2\gamma(ik)} e^{p(ik)t + p^*(ik)t'} (e^{2\gamma(ik)t} - 1) \]

\[ = \sum_{|k| \leq N} \frac{\tilde{\sigma}_k^2}{2\gamma(ik)} e^{\tilde{p}(ik)(t-t')} e^{-\gamma(ik)(t'-t)} (1 - e^{-2\gamma(ik)t}). \]

\[ \text{Var} [u(x,t)] = R(x,x,t,t) = \sum_{|k| \leq N} \frac{\tilde{\sigma}_k^2}{2\gamma(ik)} (1 - e^{-2\gamma(ik)t}), \]

Fix \( t' = t + \tau \) and let \( t \to \infty \), then the temporal correlations converge as \( t \uparrow \infty \) to the stationary correlations in physical space of the climatological state,

\[ R(\tau) = \sum_{|k| \leq N} \frac{\tilde{\sigma}_k^2}{2\gamma(ik)} e^{-\tilde{p}(ik)\tau} e^{-\gamma(ik)\tau}. \]

Similarly, the spatial correlation at fixed time in the signal is given by

\[ R(x, x', t, t) = \sum_{|k| \leq N} \frac{\tilde{\sigma}_k^2}{2\gamma(ik)} (1 - e^{-2\gamma(ik)t}) e_k(x - x'). \]
Turbulent Rossby waves

Barotropic Rossby waves with phase varying only in the north-south direction

\[
\omega_k = \frac{\beta}{k}
\]

natural to assume uniform damping
\[
\gamma(ik) = d > 0
\]
representing Ekmann friction.

known from observations that on scales of order of thousands of kilometers these waves have a k−3 energy spectrum

\[
E_k = k^{-3}.
\]

Table 5.1: Natural parameters

<table>
<thead>
<tr>
<th>mean radius of the earth</th>
<th>(a = 6.37 \times 10^6) m</th>
</tr>
</thead>
<tbody>
<tr>
<td>earth angular speed of rotation</td>
<td>(\Omega = 7.292 \times 10^{-5}) rad/sec</td>
</tr>
</tbody>
</table>
In our model, we consider a periodic domain of length $2\pi$ so that the radius is a unit length $a = 1$. Converting the time into days, we find that the parameter $\beta$ at latitude $\theta = 45^\circ$ is

$$\beta = 2\Omega \cos(45^\circ) \text{ /sec} = 8.91 \text{ /day}$$

and thus the natural frequency for this model is given by

$$\omega_k = \frac{8.91}{k}.$$ 

so the lowest wavenumber, $k = 1$, has an oscillation period of roughly 17 hours.

<table>
<thead>
<tr>
<th>number of modes</th>
<th>$K = 20$</th>
</tr>
</thead>
<tbody>
<tr>
<td>time step</td>
<td>$\Delta t = 0.01$</td>
</tr>
<tr>
<td>maximum simulation time</td>
<td>$T = 100$ days</td>
</tr>
<tr>
<td>damping time scale</td>
<td>$d = 1.5$</td>
</tr>
<tr>
<td>natural oscillation frequency</td>
<td>$\omega_k = \frac{8.9}{k}$</td>
</tr>
<tr>
<td>initial conditions</td>
<td>$u_k(0) \sim N(0, k^{-3})$</td>
</tr>
<tr>
<td>ensemble size</td>
<td>$N = 100$</td>
</tr>
<tr>
<td>noise level</td>
<td>$\tilde{\sigma}_k^2 = 2d/k^3$</td>
</tr>
</tbody>
</table>

Table 5.2: Diagnostic parameters for turbulent Rossby waves.

The parameter $d = 1.5$ is chosen such that the decorrelation time is 3 days.
Figure 5.2: Statistics of turbulent the Rossby waves: (a) mean of $u(x, t)$ as a function of space, (b) variance as a function of time (in days), (c) temporal correlation function (in days) as a function of time, and (d) spatial correlation function as function of space. In each panel, the Monte-Carlo simulated statistical quantities are denoted in solid (see text for details) and the analytical formula for the corresponding statistics in Section 5.3 is plotted in dashes.
Appendix A: Temporal Correlation Function for each Fourier mode

\[ \hat{u}_k(t) = \hat{u}_k(0)e^{p(ik)t} + \int_0^t \hat{F}_k(s)e^{p(ik)(t-s)}\,ds + \tilde{\sigma}_k \int_0^t e^{p(ik)(t-s)}dW_k(t), \]

The correlation function is defined as follows

\[ R_k(t, t') = \langle (\hat{u}_k(t) - \hat{u}_k)(\hat{u}_k(t') - \hat{u}_k)^* \rangle \]

\[ = \langle \tilde{\sigma}_k \int_0^t e^{p(ik)(t-s)}dW_k(s) \left( \tilde{\sigma}_k \int_0^{t'} e^{p(ik)(t'-s')}dW_k(s') \right)^* \rangle \]

\[ = \tilde{\sigma}_k^2 e^{p(ik)t+\rho^*(ik)t'} \int_0^t \int_0^{t'} e^{-(p(ik)s+p^*(ik)s')} \frac{1}{2} \sum_{i=1}^{2} \langle dW_{k,i}(s)dW_{k,i}(s') \rangle \]

\[ = \tilde{\sigma}_k^2 e^{p(ik)t+p^*(ik)t'} \int_0^t \int_0^{t'} e^{-(p(ik)s+p^*(ik)s')} \delta(s - s')dsds' \]

\[ = \tilde{\sigma}_k^2 e^{p(ik)t+p^*(ik)t'} \int_0^{t'} e^{-(p(ik)+p^*(ik))s'}ds' \]

\[ = \frac{\tilde{\sigma}_k^2}{2\gamma(ik)} e^{p(ik)t+p^*(ik)t'} \left( e^{2\gamma(ik)t} - 1 \right) \]

\[ = \frac{\tilde{\sigma}_k^2}{2\gamma(ik)} e^{\gamma(ik)(t-t')-\rho(ik)(t'-t)} \left( 1 - e^{-2\gamma(ik)t} \right). \]
Appendix A: Temporal Correlation Function for each Fourier mode

Let $t' = t + \tau$

$$\mathcal{R}_k(\tau) = E_k e^{-\gamma(ik)\tau} e^{-\tilde{p}(ik)\tau}.$$ 

Recall that for frequency $\tilde{p}(ik) = i\omega_k$ as in (5.3), we obtain

$$\mathcal{R}_k(\tau) = E_k e^{-\gamma(ik)\tau} [\cos(\omega_k\tau) - i\sin(\omega_k\tau)].$$
Appendix B: Spatio-Temporal Correlation Function

\[ \mathcal{R}(x, x', t, t') \equiv \langle [u(x, t) - \langle u(x, t) \rangle][u(x', t') - \langle u(x', t') \rangle]^* \rangle \]

\[ = \left\langle \sum_{|k| \leq N} \tilde{\sigma}_k \int_0^t e^{p(ik)(t-s)} e_k(x) dW_k(s) \left( \sum_{|k'| \leq N} \tilde{\sigma}_{k'} \int_0^{t'} e^{p(ik')(t'-s')} e_{k'}(x') dW_{k'}(s') \right)^* \right\rangle \]

\[ = \sum_{|k| \leq N} \tilde{\sigma}_k^2 e^{p(ik)t+p^*(ik)t'} \int_0^t \int_0^{t'} e^{-(p(ik)s+p^*(ik)s')} e_k(x-x') \frac{1}{2} \sum_{i=1}^2 \langle dW_{k,i}(s) dW_{k,i}(s') \rangle \]

\[ = \sum_{|k| \leq N} \tilde{\sigma}_k^2 e^{p(ik)t+p^*(ik)t'} \int_0^t \int_0^{t'} e^{-(p(ik)s+p^*(ik)s')} e_k(x-x') \delta(s-s') ds ds' \]

\[ = \sum_{|k| \leq N} \tilde{\sigma}_k^2 e^{p(ik)t+p^*(ik)t'} \int_0^{t'} e^{-(p(ik)+p^*(ik))s'} e_k(x-x') ds' \]

\[ = \sum_{|k| \leq N} \frac{\tilde{\sigma}_k^2}{2\gamma(ik)} e^{p(ik)t+p^*(ik)t'} (e^{2\gamma(ik)t} - 1) e_k(x-x'). \tag{5.32} \]

The third line, we use the fact that \( e_k(x) = e^{2\pi ikx} \) is an orthogonal basis and the properties of white noise in Section 2.1.1 of Chapter 2 in addition to the definition of the complex white noise \( dW_k(t) = 2^{-1/2}(dW_{k,1}(t) + idW_{k,2}(t)) \) where each \( W_{k,i}(t) \) is a Wiener process. Furthermore, we simply assume that \( \tilde{\sigma}_k = \tilde{\sigma}_k^* \).
Chapter 6: Filtering Turbulent Signals: Plentiful Observations

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The difficulties in filtering turbulent complex systems are largely due to our incomplete understanding of the dynamical system that underlies the observed signals, which have many spatio-temporal scales and rough turbulent energy spectra near the resolved mesh scale. In this chapter, we develop theoretical criteria as guidelines to address issues for filtering turbulent signals in an idealized context. In particular, we consider the simplest turbulent model discussed in Chapter 5 with plentiful observations, that is, the observations are available at every model grid point.

In this idealized context, we will provide a useful insight into answering several practical issues, including:

- As the model resolution is increased, there is typically a large computational overhead in propagating the dynamical operator and this restricts the predictions to relatively small ensemble sizes. When is it possible to filter using standard explicit and implicit solvers for the original dynamic equations by using a large time step equal to the observation time (even violating the CFL stability condition with standard explicit scheme) to increase ensemble size, yet still retain statistical accuracy?

- If plentiful observations are available on refined meshes, what is gained by increasing the resolution of the operational model? How does this depend on the nature of the turbulent spectrum?
In particular, in this simplified context, we address the basic issues outlined in 1.a)-1.d) of Chapter 1.

1.a) **Turbulent Dynamical Systems to Generate the True Signal.** The true signal from nature arises from a turbulent nonlinear dynamical system with extremely complex noisy spatio-temporal signals which have significant amplitude over many spatial scales.

1.b) **Model Errors.** A major difficulty in accurate filtering of noisy turbulent signals with many degrees of freedom is model error; the fact that the true signal from nature is processed for filtering and prediction through an imperfect model where by practical necessity, important physical processes are parameterized due to inadequate numerical resolution or incomplete physical understanding. The model errors of inadequate resolution often lead to rough turbulent energy spectra for the truth signal to be filtered on the order of the mesh scale for the dynamical system model used for filtering.

1.c) **Curse of Ensemble Size.** For forward models for filtering, the state space dimension is typically large, of order $10^4$ to $10^8$, for these turbulent dynamical systems, so generating an ensemble size with such direct approach of order 50 to 100 members is typically all that is available for real-time filtering.

1.d) **Sparse, Noisy, Spatio-Temporal Observations for only a Partial Subset of the Variables.** In systems with multiple spatio-temporal scales, the sparse observations of the truth signal might automatically couple many spatial scales, as shown below in Chapter 7 [65], while the observation of a partial subset of variables might mix together temporal slow and fast components of the system [53, 54] as discussed in Chapter 10. For example observations of pressure or temperature in the atmosphere mix slow vortical and fast gravity waves processes.
In particular, in this simplified context, we address the basic issues outlined in 1.a)-1.d) of Chapter 1. Finally, we should point out that on top of addressing the above practical issues, our novel filtering strategy reduces filtering $s$-dimensional systems in $(2N+1)^d$ resolved grid points, which involves propagating a large covariance matrix of size $s(2N+1)^d \times s(2N+1)^d$ with classical Kalman filter or extended Kalman filter, to filtering $N^d$ independently $s$-dimensional problems, each of which involves propagating a covariance matrix of size $s \times s$. We will show that the filtered solutions with this strategy are not only comparable to those using the standard ensemble Kalman filter, but they are also insensitive to the ensemble size, to the model resolution, and they are independent of any tunable parameters.
6.1 A Mathematical Theory for Fourier Filter Reduction

In Chapter 4, we reviewed the classical stability theory for a numerical solver which involves spectral analysis of a constant coefficient linearized partial differential equation. Our reduced filtering strategy is motivated by this classical stability criterion but we now employ the spectral transform not only to the PDE but also to the noisy observations (see Fig 6.1). We take advantage of the fact that the structure of our simplest turbulent model (the linear stochastic PDE, see Chapter 5) is completely decoupled into systems of Langevin equations. Together with the Fourier coefficients of the noisy observed signals, we have a reduced filtering strategy in Fourier space which we refer as the Fourier Domain Kalman Filter (FDKF). The distinct feature in this innovative strategy is that it ignores the correlations between different Fourier modes. In Chapters 7, 8, and 11, we shall see that this ignorance is in fact advantageous even for nonlinear systems with sparse regular observations. The mathematical theory developed here allows detailed Fourier analysis, akin to the difference scheme stability analysis of Chapter 4, to provide guidelines for filtering turbulent signals for stochastic PDE’s provided the observations are plentiful. The parallel theory for sparse observations is described in Chapter 7.

To be more precise, consider the following canonical filtering problem in the real domain. The canonical stochastic PDE in (6.1) is an $s \times s$ dynamical system with a Gaussian spatially correlated $s \times s$ random field white noise forcing $\sigma(x)\dot{\tilde{W}}(t)$, constructed so that the observed climatological turbulent energy spectrum is reproduced as the stationary invariant measure of (6.1).
6.1 A Mathematical Theory for Fourier Filter Reduction

**Figure 6.1**: Fourier Domain Kalman Filter (FDKF) blends the classical stability analysis of numerical PDE and the classical spatial domain Kalman filter.
6.1 A Mathematical Theory for Fourier Filter Reduction

Canonical Filtering Problem: Plentiful Observations

\[
\frac{\partial}{\partial t} \bar{u}(x, t) = P \left( \frac{\partial}{\partial x} \right) \bar{u}(x, t) - \gamma \left( \frac{\partial}{\partial x} \right) \bar{u}(x, t) + \sigma(x) \dot{W}(t),
\]

\[\bar{u}(x, 0) = \bar{u}_0,\]

\[\bar{v}(x_j, t_m) = G\bar{u}(x_j, t_m) + \bar{\eta}_{j,m}.
\]

As in standard finite difference linear stability analysis, the problem in (6.1) is nondimensionalized to a $2\pi$--periodic domain so that continuous and discrete Fourier series can be used to analyze (6.1) and the related discrete approximations. In our canonical test problem, we realize the PDE (6.1) at $2N + 1$ discrete points \( \{x_j = jh, j = 0, \ldots, N\} \) such that \( (2N+1)h = 2\pi \). This one-space dimension problem is chosen for simplicity in exposition but without lost of generality one can generalize it to \( d \)-space dimensions.
6.1 A Mathematical Theory for Fourier Filter Reduction

We consider $q \leq s$ observations $\bar{u}(x_j, t_m)$ in (6.3) which are attainable at every discrete time $t_m$ and at every model grid point $x_j$ with a fixed $q \times s$ observation matrix $G$. These plentiful observations are assumed to be imprecise, that is, they contain random measurement errors represented by zero mean Gaussian random variables $\sigma^o_m = \{\sigma^o_{j,m}\}$ that are spatially and temporally independent at different grid points with covariance matrix

$$\langle \sigma^o_m \otimes (\sigma^o_m)^* \rangle = R^o,$$

where $R^o$ is a block diagonal observation error covariance matrix with $q \times q$ block diagonal component $r^o I$.

Recall the finite Fourier expansion as in Chapter 4,

$$\bar{u}(x_j, t_m) = \sum_{|k| \leq N} \bar{u}_k(t_m)e^{ikx_j}, \quad \hat{u}_k = \hat{u}^*_k$$

(6.5)

$$\bar{u}_k(t_m) = \frac{h}{2\pi} \sum_{j=0}^{2N} \bar{u}(x_j, t_m)e^{-ikx_j}.$$  

(6.6)

Substituting (6.5) into (6.1)-(6.3) and using the identity (6.6), we obtain the following $s$-dimensional canonical filtering problem with $q$ observations for each Fourier mode.
Fourier Analogue of the Canonical Filtering Problem: Plentiful Observations

\[ \tilde{u}_k(t_m + 1) = F_k \tilde{u}_k(t_m) + \tilde{\sigma}_{k,m+1}, \]  
\[ \tilde{u}_k(t_0) = \tilde{\mu}_k, \]  
\[ \tilde{v}_k(t_m) = G\tilde{u}_k(t_m) + \tilde{\sigma}_{k,m}^o. \]  

(6.7)  
(6.8)  
(6.9)

In (6.7), operator \( F_k \) is an \( s \times s \) diagonal matrix that solves or approximates the deterministic part of \( s \)-dimensional Langevin equation in (5.8). In (6.7), the zero mean complex Gaussian noises, \( \tilde{\sigma}_{k,m} \), are uncorrelated in time and their second moment is given by

\[ \langle \tilde{\sigma}_{k,m} \otimes (\tilde{\sigma}_{k',m})^* \rangle = \delta_{k-k'} R_k, \quad |k|, |k'| \leq N, \]  

(6.10)

with \( R_k \) a strictly positive definite covariance matrix; for a vector Langevin equation, \( R_k \) is a diagonal \( s \times s \) matrix.

In Fourier space, however, the observation noises have to be derived in careful manner. Specifically, the observation error covariance is inversely proportional to the mesh size. In
6.1 A Mathematical Theory for Fourier Filter Reduction

In particular,

\[
\langle \tilde{\sigma}_k^{o}, m \otimes (\tilde{\sigma}_k^{o}, m)^* \rangle = \left\langle \frac{\hbar}{2\pi} \sum_{j=0}^{2N} \tilde{\sigma}_{j,m} e^{-ikx_j} \otimes \left( \frac{\hbar}{2\pi} \sum_{j'=0}^{2N} \tilde{\sigma}_{j',m} e^{-i k' x_j} \right)^* \right\rangle
\]

\[
= \frac{\hbar^2}{(2\pi)^2} \sum_{j=0}^{2N} \langle \tilde{\sigma}_{j,m} \otimes \tilde{\sigma}_{j,m}^* e^{-i(k-k')x_j} \rangle
\]

\[
= \frac{\hbar^2}{(2\pi)^2} \sum_{j=0}^{2N} \langle \tilde{\sigma}_{j,m} \otimes \tilde{\sigma}_{j,m}^* \rangle \langle e^{-i(k-k')x_j} \rangle
\]

\[
= \frac{\hbar^2}{(2\pi)^2} \sum_{j=0}^{2N} r^o I \delta_{k-k'}
\]

\[
= \frac{r^o I}{2N + 1} \delta_{k-k'}.
\]

(6.11)

Here, we use the independence of observations at different locations as in (6.4) and the fact that \( \{e^{ikx_j}\} \) form a basis in \( k \)-space (see Chapter 4).

Consequently, we have the following [100]:

6.1 A Mathematical Theory for Fourier Filter Reduction

**Theorem 6.1.** If the observation points coincide with the discrete mesh points then for both the truth and any finite difference approximation:

- If the covariance matrix for the initial data, \( \tilde{u}_{k,0} \), has the same block diagonal structure as in Eqns (6.10) and (6.11) for the system and observation noise, i.e., different Fourier modes are uncorrelated for \( k \geq 0 \), then the Kalman filtering test problem is equivalent to studying the independent \( s \times s \) matrix Kalman filtering problems in Eqns (6.7)-(6.11).

- Provided that the \( s \times s \) independent Kalman filtering problems in Eqns (6.7)-(6.11) are observable (see Chapters 2 and 3), then the unique steady-state limiting Kalman filter factors for the complete model into a block diagonal product of the limiting Kalman filters for each individual wave number, \( k \).

The practical significance of this result is that off-line tests for filter stability and model error for extremely complex PDE’s can be developed for the simpler \( s \times s \) matrix problems. For systems with \( s > 1 \) and observation matrix \( G \) with \( q < s \) observations at each grid point, we are in the situation of filtering fewer observations than the actual dimension of the variables; this situation readily arises for the shallow water equation or geophysical primitive equations [95, 116] where only the pressure and temperature might be known at each observation point.
6.1 The Number of Observation Points Equals the Number of Discrete Mesh Points: Mathematical Theory

In the above analysis, we studied the simplest situation where given $2N + 1$ discrete mesh points for the dynamical operator, $x_j = jh$, $j = 0, 1, \ldots, 2N$, the $2N + 1$ observation points, $\tilde{x}_j$, in (6.3) exactly coincide with these mesh points. Here we consider the situation where the $2N + 1$ observation points, $\tilde{x}_j$, do not coincide with the mesh point as often occurs in some applications, i.e., there are more observations over one area and fewer observations over another area. Here we present the mathematical theory [100] that establishes that suitable Fourier diagonal filters as in (6.7)-(6.9) provide upper and lower bounds on filter performance in the present setting. Lemma's 3.1 and 3.2 from Chapter 3 will be used in obtaining the rigorous upper and lower bound estimates.

For $\tilde{x}_j \neq \tilde{x}_k$, $0 \leq j, k \leq 2N$, consider the non-equispaced scalar real valued trigonometric interpolation problem,

\[
\begin{align*}
  f(\tilde{x}_j) & = f_j, \quad j = 0, 1, \ldots, 2N, \\
  f(x) & = \sum_{|k| \leq N} \hat{f}_k e^{ikx}.
\end{align*}
\]  

(6.12)
6.1 The Number of Observation Points Equals the Number of Discrete Mesh Points: Mathematical Theory

Write $\hat{f}_k = a_k + ib_k$, and the vector in $\mathbb{R}^{2N+1}$, $(a_0, a_1, b_1, \ldots, a_N, b_N)$ uniquely determines the finite Fourier series. For the $2N + 1$ distinct points $\tilde{x}_j$, there is a unique invertible mapping $V : \mathbb{R}^{2N+1} \to \mathbb{R}^{2N+1}$ which solves the above trigonometric interpolation problem; with $\tilde{f} = (f_0, \ldots, f_{2N})$,

$$V \tilde{f} = (a_0, a_1, b_1, \ldots, a_N, b_N)^T. \quad (6.13)$$

There are even well-known explicit classical numerical analysis formulas for this unequally spaced trigonometric interpolation problem in a single space variable that define $V$ (see Chapter 5 of [71]). Obviously, the fixed matrix multiplication $G = \{G_{ij}\} \in \mathbb{R}^{s \times s}$ that defines the observations still commutes with the scalar transformation matrix, $V$, i.e.,

$$V \sum_{j=1}^{s} G_{ij} \tilde{u}_j(t_m) = \sum_{j=1}^{s} G_{ij} V \tilde{u}_j(t_m), \quad i = 1, \ldots, s$$

$$= \sum_{j=1}^{s} G_{ij} (a_{j,0}(t_m), a_{j,1}(t_m), b_{j,1}(t_m), \ldots, a_{j,N}(t_m), b_{j,N}(t_m))^T \quad (6.14)$$

where $\tilde{u}_j(t_m) \in \mathbb{R}^{2N+1}$ is the $j$th variable of the true state at $2N + 1$ discrete points and $(a_{j,0}, a_{j,1}, b_{j,1}, \ldots, a_{j,N}, b_{j,N})$ is as defined in (6.13). Applying the interpolation formulas defined in (6.12) and (6.13) to the canonical observation (6.3), collecting the $k$th wave number from identity (6.14), and using the fact that $\tilde{u}_{i,k} = a_{i,k} + ib_{i,k}$, we have an $s$-dimensional canonical observation for each wave number $k$:

$$\tilde{v}_k(t_m) = (V \tilde{v}_1(t_m), \ldots, V \tilde{v}_s(t_m))_k = G \tilde{u}_k(t_m) + (V \tilde{\sigma}_{m,1}, \ldots, V \tilde{\sigma}_{m,s})_k. \quad (6.15)$$
6.1 The Number of Observation Points Equals the Number of Discrete Mesh Points: Mathematical Theory

The formula in (6.15) can be continued trivially to $k$ with $-N \leq k < 0$ by using the conjugate formula $\hat{u}_{-k} = \hat{u}_k^*$. In the present case, the transformed observation noise for each $i = 1, \ldots, s$ in general has correlations across different Fourier modes with correlation matrix

$$
\langle V \bar{\sigma}_{m,i} \otimes (V \bar{\sigma}_{m,i})^T \rangle = V \langle \bar{\sigma}_{m,i} \otimes (\bar{\sigma}_{m,i})^T \rangle V^T
= r^o V V^T.
$$

(6.16)

The full observation noise correlation matrix is an $s(2N + 1) \times s(2N + 1)$ matrix consists of spatially correlated block diagonals (6.16) since the observation noise for each variable in the $s$-dimensional system is assumed to be uncorrelated. Therefore, the upper and lower bounds of the positive definite correlation matrix in (6.16),

$$
r^o c_-^2 I_{2N+1} \geq r^o V V^T \geq r^o c_+^2 I_{2N+1}
$$

(6.17)

with $c_+^2 > c_-^2 > 0$ depending on $N$, is also the upper and lower bounds of the full correlation matrix. Now consider the two Fourier diagonal problems with observations,

$$
\tilde{u}_k(t_m) = G \tilde{u}_k(t_m) + \bar{\sigma}_{m,k}^\pm.
$$

(6.18)

with the covariance given by

$$
\langle \bar{\sigma}_{m,k}^\pm \otimes (\bar{\sigma}_{m,k}^\pm)^* \rangle = r^o c_\pm^2 I_s.
$$

(6.19)

With (6.17) and Lemma's 3.1, 3.2 which guarantee the monotonicity of the Kalman filter covariance with respect to the observation noise, we immediately have:
6.1 The Number of Observation Points Equals the Number of Discrete Mesh Points: Mathematical Theory

Lemma 3.1. Consider the map on positive definite covariance matrices

\[ \Phi(P, R) = P - P(G^+)^T(G^+ P(G^+)^T + R)^{-1}G^+ P, P \in \mathbb{R}^{N_+ \times N_+}. \] (3.39)

Then

\[ P_1 \leq P_2 \Rightarrow \Phi(P_1, R) \leq \Phi(P_2, R), \]

and

\[ 0 < R_1 \leq R_2 \Rightarrow \Phi(P, R_1) \leq \Phi(P, R_2). \]

Lemma 3.2. Consider the dynamic map on positive definite covariance matrices

\[ \Psi(P, R) = F_+ P F_+^T + R, P \in \mathbb{R}^{N_+ \times N_+}. \] (3.40)

Then

\[ P_1 \leq P_2 \Rightarrow \Psi(P_1, R) \leq \Psi(P_2, R), \]

and

\[ 0 < R_1 \leq R_2 \Rightarrow \Psi(P, R_1) \leq \Psi(P, R_2). \]
Theorem 6.2. If there are $2N + 1$ observation points $\tilde{x}_j$, which do not coincide with the grid points, then upper and lower bounds on the discrete Kalman filtering process or the truth model as in Theorem 6.1 are achieved through the independent decoupled $s \times s$ filtering problems for each wave number $k$ defined in (6.7) and (6.18) with the upper- and lower-bound diagonal noise covariances in (6.19) involving $c^2_\pm$.

While Theorem 6.2 is interesting, it is often not practically useful since the condition numbers in (6.17) of the non-equidistant discrete Fourier transform in (6.12) and (6.13) have $c^2_+/c^2_- \gg 1$ and this inversion problem is ill-conditioned. It is actually much better to use linear interpolation of the observations to a regular grid; surprisingly, it is a bad idea to use smoother interpolation algorithms for turbulent signals [61].