Systematic strategies for real time filtering of turbulent signals in complex systems

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Modern Applied Modus Operandi

**Theory:** Important mathematical guidelines
Qualitative Exactly Solvable Models

**Novel Algorithms:**
Applications to Real Problems in Science/Engineering

**General Refs for Talk:** Research/Expository

**Introductory Graduate Text**
Filtering...

- ...is the process of obtaining the best statistical estimate of a natural system from partial observations of the true signal from nature.

- ... weather and climate in real time as well as the spread of plumes or pollutants.

- ... is a hard problem especially when the model resolution is increased.

- ... of weather and climate usually involves extremely unstable, chaotic dynamical systems with many spatio-temporal scales and rough turbulent energy spectra.

Filtering low-dimensional systems

- For low-dimensional systems or systems with a low-dimensional attractor Monte-Carlo approaches are very efficient.

- For example: Particle-filter schemes provide very good estimates even in the presence of strong-nonlinearity and highly-non Gaussian distributions.

- For problems of high-dimensionality the computational cost for particle-filter methods is prohibited.

- However, some progress have been made on this direction with particle filters with small ensembles that work for turbulent signals (Ch. 15).
Filtering high-dimensional systems

In general we have two paths:

*Maximum Entropy Particle Filter*

- Judicious use of partial marginal distributions to avoid particle collapse.

*Bayesian hierarchical modeling and reduced order filtering*

- Based on Kalman filter.
- Work even for extremely complex high dimensional systems.

Good results for synoptic scale mid-latitude weather dynamics

Important sensitivity to model resolution, observation frequency and the nature of the turbulent signal.

Less skillful for more complex coupled phenomena such gravity waves coupled with condensational heating from clouds (important for the tropics and severe local weather)
Fundamental challenges for real-time filtering of turbulent signals

- Turbulent dynamical systems to generate the true signal.

- Model errors (post-processing of measured signal through imperfect models, inadequate measurement resolution).

- Course of ensemble size. (state space dimension of order $10^4$-$10^8$ allows only for a small ensemble size: 50-100).

- Sparse, noisy, spatio-temporal observations for only a partial set of variables.
Challenging Questions (1/2)

• How to develop simple off-line mathematical test criteria as guidelines for filtering extremely stiff multiple space-time scale problems that often arise in filtering turbulent signals through plentiful and sparse observations?

• For turbulent signals from nature with many scales, even with mesh refinement the model has inaccuracies from parametrization, under-resolution, etc. Can judicious model error help filtering and simultaneously overcome the curse of dimension?
Challenging Questions (2/2)

• Can new computational strategies based on stochastic parameterization algorithms be developed to overcome the curse of dimension, to reduce model error and improve the filtering as well as the prediction skill?

• Can exactly solvable models be developed to elucidate the central issue of sparse, noisy, observations for turbulent signals, to develop unambiguous insight into model errors, and to lead to efficient new computational algorithms?
Turbulent dynamical systems and basic filtering

- Fundamental statistical differences of high- and low-dimensional chaotic dynamical systems.

- L63 is a low-dimensional chaotic dynamical system which is weakly mixing with one unstable direction.

- L96 model is an example of a prototype turbulent dynamical system

\[ \frac{du_j}{dt} = (u_{j+1} - u_{j-2})u_{j-1} - u_j + F, \quad j = 0, \ldots, J - 1, \]

- Designed to mimic baroclinic turbulence in the midlatitude atmosphere with the effects of energy conserving non-linear advection and dissipation.
The Lorenz 96 model

- Depending on the forcing value $F$ the system will exhibit completely different dynamical features

![Image of space-time plots for different $F$ values](image)

Figure 1.1: Space-time of numerical solutions of L-96 model for weakly chaotic ($F = 6$), strongly chaotic ($F = 8$), and fully turbulent ($F = 16$) regime.

Table 1.1: Dynamical properties of L-96 model for regimes with $F = 6, 8, 16$. $\lambda_1$ denotes the largest Lyapunov exponent, $N^+$ denotes the dimension of the expanding subspace of the attractor, $KS$ denotes the Kolmogorov-Sinai entropy, and $T_{corr}$ denotes the decorrelation time of energy-rescaled time correlation function.

<table>
<thead>
<tr>
<th>Regime</th>
<th>$F$</th>
<th>$\lambda_1$</th>
<th>$N^+$</th>
<th>$KS$</th>
<th>$T_{corr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weakly chaotic</td>
<td>6</td>
<td>1.02</td>
<td>12</td>
<td>5.547</td>
<td>8.23</td>
</tr>
<tr>
<td>Strongly chaotic</td>
<td>8</td>
<td>1.74</td>
<td>13</td>
<td>10.94</td>
<td>6.704</td>
</tr>
<tr>
<td>Fully turbulent</td>
<td>16</td>
<td>3.945</td>
<td>16</td>
<td>27.94</td>
<td>5.594</td>
</tr>
</tbody>
</table>
The two layer QG model

- Double periodic geometry
- Externally forced by a mean vertical shear
- Presence of baroclinic instability

- Governing equations for a flat bottom, equal depth layers, and rigid rid

\[
\frac{\partial q_1}{\partial t} + J(\psi_1, q_1) + U \frac{\partial q_1}{\partial x} + (\beta + k_d^2 U) \frac{\partial \psi_1}{\partial x} + \nu \nabla^8 q_1 = 0,
\]
\[
\frac{\partial q_2}{\partial t} + J(\psi_2, q_2) - U \frac{\partial q_2}{\partial x} + (\beta - k_d^2 U) \frac{\partial \psi_2}{\partial x} + \kappa \nabla^2 \psi_2 + \nu \nabla^8 q_2 = 0,
\]

with \( q \) being the perturbed QG potential vorticity

\[
q_i = \beta y + \nabla^2 \psi_i + \frac{k_d^2}{2} (\psi_{3-i} - \psi_i), \quad i = 1, 2,
\]

\( k_d = \sqrt{8/L_d} \) is the wavenumber corresponding to the Rossby radius \( L_d \)
\( \kappa \) is the Ekman bottom drag coefficient
\( \nu \) is the hyperviscosity coefficient
The two layer QG model

- It is the simplest climate model for the poleward transport of heat in the atmosphere or ocean.

- Resolution of 128x128x2 has a phase space of 30000 variables

- To model atmosphere or ocean this model is turbulent with strong energy cascade.

- It has been recently utilized as a test model for filtering algorithms in the atmosphere and ocean.
Basic Filtering

1. Forecast (Prediction)

2. Analysis (Correction)

Figure 1.2: Filtering: Two-steps predictor corrector method.

Filtering is a two-step process involving statistical prediction of the state variables through a forward operator followed by an analysis step at the next observation time which corrects this prediction on the basis of the statistical input of noisy observations of the system.
Example of application: predicting path of hurricane
Basic Filtering

- In the present applications, the forward operator is a large dimensional dynamical system usually written in the Ito sense

\[ \frac{du}{dt} = F(u, t) + \sigma(u, t) \dot{W}(t) \]

for \( u \in \mathbb{R}^N \), where \( \sigma \) is an \( N \times K \) noise matrix and \( \dot{W} \in \mathbb{R}^K \) is \( K \)-dimensional white noise.

- This dynamical system can also be written as a transport equation for the pdf

\[
\begin{align*}
p_t &= -\nabla_u \cdot (F(u, t)p) + \frac{1}{2} \nabla_u \cdot \nabla_u (Qp) \equiv L_{FP} p \\
p_t|_{t=t_0} &= p_0(u) \quad \text{with } Q(t) = \sigma \sigma^T
\end{align*}
\]
Basic Filtering - dynamics

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\text{with } Q(t) &= \sigma\sigma^T
\end{align*}
\]
Basic Filtering - observations

- We assume M linear observations of the true signal from nature

\[ \bar{v}_m = G u(m \Delta t) + \bar{\sigma}_m^o, \quad m = 1, 2, \ldots \]

where \( G \) maps \( \mathbb{R}^N \) into \( \mathbb{R}^M \) while the observational noise, \( \bar{\sigma}_m^o \in \mathbb{R}^M \), is assumed to be a zero mean Gaussian random variable with \( M \times M \) covariance matrix,

\[ R^o = \langle \bar{\sigma}_m^o \otimes (\bar{\sigma}_m^o)^T \rangle. \]

Gaussian random variables are uniquely determined by their mean and covariance.
Algorithmic description of filtering

1. We start at time $m \Delta t$ with a posterior distribution $p_{m,+}(u)$.

2. Calculate, using FP equation, a prediction or forecast probability distribution, $p_{m+1,-}(u)$

   $$ p_t = L_{FP} p, \quad m \Delta t < t \leq (m + 1) \Delta t $$

   $$ p|_{t=m\Delta t} = p_{m,+}(u). $$

3. Next we have the analysis step which corrects this forecast taking into account observations (using Bayes Thm)

   $$ p_{m+1,+}(u) \equiv p_{m+1}(u|v_{m+1}) = \frac{p_{m+1}(v_{m+1}|u)p_{m+1,-}(u)}{\int p_{m+1}(v_{m+1}|u)p_{m+1,-}(u) \, du}. $$
The Kalman Filter

- Linear dynamics between observations. This yields the forward operator

\[ u_{m+1} = F u_m + \bar{f}_{m+1} + \sigma_{m+1}. \]

Here \( F \) is the \( N \times N \) system operator matrix and \( \sigma_m \) is the system noise assumed to be zero mean and Gaussian with \( N \times N \) covariance matrix

\[ R = \langle \sigma_m \otimes \sigma_m^T \rangle, \forall m, \]

while \( \bar{f}_m \) is a deterministic forcing.

- Gaussian initial conditions

\[ p_0(u) = \mathcal{N}(\bar{u}_0, R_0) \]

By recursion:

\[ p_{m,+}(u) = \mathcal{N}(\bar{u}_{m,+}, R_{m,+}) \]
The Kalman Filter

• Forecast or prediction step using linear dynamics is also Gaussian

\[ p_{m+1,-}(u) = \mathcal{N}(\tilde{u}_{m+1,-}, R_{m+1,-}) \]
\[ \tilde{u}_{m+1,-} = F\tilde{u}_{m,+} + \bar{f}_{m+1} \]
\[ R_{m+1,-} = FR_{m,+}F^T + R. \]

• Using the assumption of linear dynamics and Gaussian statistics the analysis step becomes an explicit regression procedure for Gaussian random variables yielding the Kalman filter

\[ p_{m+1,+}(u) = \mathcal{N}(\tilde{u}_{m+1,+}, R_{m+1,+}) \]
\[ \tilde{u}_{m+1,+} = (I - K_{m+1}G)\tilde{u}_{m+1,-} + K_{m+1}v_{m+1} \]
\[ R_{m+1,+} = (I - K_{m+1}G)R_{m+1,-} \]
\[ K_{m+1} = R_{m+1,-}G^T(GR_{m+1,-}G^T + R^o)^{-1}. \]
The Kalman Filter

- The posterior mean is a weighted sum of the forecast and analysis distributions through the Kalman gain matrix.

- The observations always reduce the total covariance.

- In the Gaussian case with linear observations, the analysis step is a standard linear least squares regression.

- For linear systems without model errors, the recursive Kalman filter is an optimal estimator – this is not always the case for nonlinear systems.
Goal: Provide math guidelines and new numerical strategies thru modern applied math paradigm

**Modelling Turbulent Signals**
- Stochastic Langevin Models
- Complex Nonlinear Dynamical Systems

**Filtering**
- Extended Kalman Filter
- Classical Stability Criteria: Observability, Controllability

**Numerical Analysis**
- Classical Von-Neumann stability analysis for frozen coefficient linear systems
PART I: Filtering Linear Problem

Real Space

Simplest Turbulent Model
Constant Coefficient
Linear Stochastic PDE

Classical Kalman Filter

Noisy Observations

Fourier Space

Independent Fourier Coefficient:
Langevin equation

Ensemble Kalman Filter

FT

Fourier Domain Kalman Filter

Innovative Strategy

Fourier Coefficients of the noisy observations

FT