

# STABLE RATIONALITY OF QUADRIC SURFACE BUNDLES OVER SURFACES

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ABSTRACT. We study rationality properties of quadric surface bundles over the projective plane. We exhibit families of smooth projective complex fourfolds of this type over connected bases, containing both rational and non-rational fibers.

## 1. INTRODUCTION

Is a deformation of a smooth rational (or irrational) projective variety still rational (or irrational)? The main goal of this paper is to show that rationality is not deformation invariant for families of smooth complex projective varieties of dimension four. Examples along these lines are known when singular fibers are allowed, e.g., smooth cubic threefolds (which are irrational) may specialize to cubic threefolds with ordinary double points (which are rational), while smooth cubic surfaces (which are rational) may specialize to cones over elliptic curves. Totaro shows that specializations of rational varieties need not be rational in higher dimensions if mild singularities are allowed [Tot15].

Let  $S$  be a smooth projective rational surface over the complex numbers with function field  $K = \mathbb{C}(S)$ . A *quadric surface bundle* consists of a fourfold  $X$  and a flat projective morphism  $\pi : X \rightarrow S$  such that the generic fiber  $Q/K$  of  $\pi$  is a smooth quadric surface. We assume that  $\pi$  factors through the projectivization of a rank four vector bundle on  $S$  such that the fibers are (possibly singular) quadric surfaces; see Section 3 for relevant background.

**Theorem 1.** *There exist smooth families of complex projective fourfolds  $\phi : \mathcal{X} \rightarrow B$  over connected varieties  $B$ , such that for every  $b \in B$  the fiber  $\mathcal{X}_b = \phi^{-1}(b)$  is a quadric surface bundle over  $\mathbb{P}^2$ , and satisfying:*

- (1) *for very general  $b \in B$  the fiber  $\mathcal{X}_b$  is not stably rational;*

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- (2) *the set of points  $b \in B$  such that  $\mathcal{X}_b$  is rational is dense in  $B$  for the Euclidean topology.*

Concretely, we consider smooth hypersurfaces

$$X \subset \mathbb{P}^2 \times \mathbb{P}^3$$

of bidegree  $(2, 2)$ ; projection onto the first factor gives the quadric surface bundle.

Our approach has two key elements. First, we apply the technique of the decomposition of the diagonal [Voi15b, CTP16, CTP15, Tot16] to show that very general  $X \subset \mathbb{P}^2 \times \mathbb{P}^3$  of bidegree  $(2, 2)$  fail to be stably rational. The point is to identify a degenerate quadric surface fibration, with non-trivial second unramified cohomology and mild singularities. The analogous degenerate conic bundles over  $\mathbb{P}^2$  are the Artin-Mumford examples; deforming these allows one to show that very general conic bundles over  $\mathbb{P}^2$  with large degeneracy divisor fail to be stably rational [HKT15]. Second, quadric surface bundles are rational over the base whenever they admit a section, indeed, whenever they admit a multisection of odd degree. If the base is rational then the total space is rational as well; this can be achieved over a dense set of the moduli space [Has99, Voi15a]. This technique also yields rationality for a dense family of cubic fourfolds containing a plane; no cubic fourfolds have been shown not to be stably rational.

This paper is inspired by the approach of Voisin [Voi15a], who also considers fourfolds birational to quadric surface bundles. While our proof of rationality is similar, the analysis of unramified cohomology relies on work of Pirutka [Pir16] and Colliot-Thélène and Ojanguren [CTO89].

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## 2. GENERALITIES

We recall implications of the “integral decomposition of the diagonal and specialization” method, following [CTP16], [Voi15b], and [Pir16].

A projective variety  $X$  over a field  $k$  is *universally  $\mathrm{CH}_0$ -trivial* if for all field extensions  $k'/k$  the natural degree homomorphism from the Chow group of zero-cycles

$$\mathrm{CH}_0(X_{k'}) \rightarrow \mathbb{Z}$$

is an isomorphism. Examples include smooth  $k$ -rational varieties. More complicated examples arise as follows:

**Example 2.** [CTP15, Lemma 2.3, Lemma 2.4] Let  $X = \cup_i X_i$  be a projective, reduced, geometrically connected variety over a field  $k$  such that:

- Each component  $X_i$  is geometrically irreducible and  $k$ -rational, with isolated singularities.
- Each intersection  $X_i \cap X_j$  is either empty or has a zero-cycle of degree 1.

Then  $X$  is universally  $\text{CH}_0$ -trivial.

A projective morphism

$$\beta : \tilde{X} \rightarrow X$$

of  $k$ -varieties is *universally  $\text{CH}_0$ -trivial* if for all extensions  $k'/k$  the push-forward homomorphism

$$\beta_* : \text{CH}_0(\tilde{X}_{k'}) \rightarrow \text{CH}_0(X_{k'})$$

is an isomorphism.

**Proposition 3.** [CTP16, Proposition 1.8] *Let*

$$\beta : \tilde{X} \rightarrow X$$

*be a projective morphism such that for every scheme point  $x$  of  $X$ , the fiber  $\beta^{-1}(x)$ , considered as a variety over the residue field  $\kappa(x)$ , is universally  $\text{CH}_0$ -trivial. Then  $\beta$  is universally  $\text{CH}_0$ -trivial.*

For example, if  $X$  is a smooth projective variety and

$$\beta : \text{Bl}_Z(X) \rightarrow X$$

is a blowup of a smooth subvariety  $Z \subset X$ , then  $\beta$  is a universally  $\text{CH}_0$ -trivial morphism, since all fibers over (scheme) points are projective spaces. More interesting examples arise as resolutions of singularities of certain singular projective varieties.

Examples of failure of universal  $\text{CH}_0$ -triviality are given by smooth projective varieties  $X$  with nontrivial Brauer group  $\text{Br}(X)$ , or more generally, by varieties with nontrivial higher unramified cohomology [CTP16, Section 1]. The following specialization argument is the key to recent advances in investigations of stable rationality:

**Theorem 4.** [Voi15b, Theorem 2.1], [CTP16, Theorem 2.3] *Let*

$$\phi : \mathcal{X} \rightarrow B$$

be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point  $b \in B$  such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

- the group  $H_{nr}^2(\mathbb{C}(X), \mathbb{Z}/2)$  is nontrivial;
- $X$  admits a desingularization

$$\beta : \tilde{X} \rightarrow X$$

such that the morphism  $\beta$  is universally  $\text{CH}_0$ -trivial.

Then a very general fiber of  $\phi$  is not stably rational.

### 3. QUADRIC SURFACE BUNDLES

Let  $S$  be a smooth projective variety over  $\mathbb{C}$ . Suppose that  $\pi : X \rightarrow S$  is a quadric surface bundle, i.e., a flat projective morphism from a variety such that the generic fiber  $Q$  is a smooth quadric surface. We assume it admits a factorization

$$X \hookrightarrow \mathbb{P}(V) \rightarrow S,$$

where  $V \rightarrow S$  is a rank four vector bundle and the fibers of  $\pi$  are expressed as quadric surfaces in the fibers of  $\mathbb{P}(V) \rightarrow S$ . There is a well-defined degeneracy divisor  $D \subset S$  corresponding to where the associated quadratic form drops rank.

Trivializing  $V$  over an open neighborhood of  $S$ ,  $X$  may be expressed using a symmetric  $4 \times 4$  matrix  $(a_{ij})$ :

$$\sum a_{ij} x_i x_j = 0.$$

The local equation for  $D$  is the determinant  $\det((a_{ij}))$ . Note that  $D$  has multiplicity  $\geq 2$  where the rank of fibers is less than three. Indeed, the hypersurface

$$\{\det(a_{ij}) = 0\} \subset \mathbb{P}_{(a_{ij})}^9$$

is singular precisely where all the  $3 \times 3$  minors vanish.

**3.1. Rationality of quadric bundles.** It is well known that  $Q$  is rational over  $K = \mathbb{C}(S)$  if and only if  $Q(K) \neq \emptyset$ , i.e., when  $\pi$  admits a rational section. A theorem of Springer [Spr52] implies that  $Q(K) \neq \emptyset$  provided  $Q(K') \neq \emptyset$  for some extension  $K'/K$  of odd degree, i.e., when  $\pi$  admits a rational multisection of odd degree. Thus we obtain

**Proposition 5.** *Let  $\pi : X \rightarrow S$  be a quadric surface bundle as above, with  $S$  rational. Then  $X$  is rational provided  $\pi$  admits a multisection of odd degree.*

Let  $F_1(X/S) \rightarrow S$  denote the relative variety of lines of  $\pi$ . Let  $S_\circ \subset S$  denote the largest open subset such that  $S_\circ \cap D$  is smooth and  $X_\circ = X \times_S S_\circ$ . Then  $F_1(X_\circ/S_\circ) \rightarrow S_\circ$  factors

$$F_1(X_\circ/S_\circ) \xrightarrow{p} T_\circ \rightarrow S_\circ,$$

where the second morphism is a double cover branched along  $S_\circ \cap D$  and the first morphism is an étale  $\mathbb{P}^1$ -bundle. In particular  $F_1(X_\circ/S_\circ)$  is non-singular. Let  $\alpha \in \text{Br}(T_\circ)[2]$  denote the Brauer class arising from  $p$ .

Let  $F$  be a resolution of the closure of  $F_1(X_\circ/S_\circ)$  in  $F_1(X/S)$  obtained by blowing up over the complement of  $S_\circ$ . The incidence correspondence between  $X$  and  $F_1(X/S)$

$$\Gamma' \subset X \times_S F_1(X/S)$$

induces a correspondence  $\Gamma$  between  $X$  and  $F$  and a homomorphism

$$\Gamma_* : \text{CH}^2(X) \rightarrow \text{Pic}(F).$$

Let  $\eta$  denote the generic point of  $S$ ; there is a homomorphism

$$\Xi_* : \text{Pic}(F_\eta) \rightarrow \text{CH}^2(X_\eta).$$

constructed as follows: Consider  $Z \subset F_\eta$  a finite reduced subscheme with support on each component of  $F_\eta$ , e.g., a choice of  $n$  lines from each ruling. Take the union of the corresponding rulings in  $X_\eta$  and set  $\Xi_*(Z) \subset X_\eta$  to be its singular locus, e.g.,  $n^2$  points where the rulings cross. This is compatible with rational equivalence and yields the desired homomorphism. Thus a divisor with odd degree on each geometric component of  $F_\eta$  gives rise to a multisection of odd degree.

The correspondences  $\Gamma$  and  $\Xi$  guarantee the following conditions are equivalent:

- $\alpha = 0$ ;
- $F$  admits a divisor intersecting the generic fiber  $F_\eta$  with odd degree on each component;
- $X$  admits a multisection of odd degree.

The correspondence  $\Gamma$  also acts at the level of Hodge classes; here we obtain an equivalence:

- $F$  admits an integral  $(1, 1)$ -class intersecting the generic fiber  $F_\eta$  with odd degree on each component;

- $X$  admits an integral  $(2, 2)$ -class intersecting the fibers of  $\pi$  with odd degree.

Applying the Lefschetz  $(1, 1)$  Theorem to  $F$  and Proposition 5 we obtain:

**Proposition 6.** *Let  $\pi : X \rightarrow S$  be a quadric surface bundle as above, with  $X$  smooth and  $S$  rational. Then  $X$  is rational if it admits an integral  $(2, 2)$ -class meeting the fibers of  $\pi$  in odd degree.*

**Remark 7.** See [CTV12, Cor. 8.2] for results on the integral Hodge conjecture for quadric bundles over surfaces; these suffice for our application to quadric surface bundles over  $\mathbb{P}^2$ .

**3.2. A key example.** The generic fiber of  $\pi$  is a quadric surface, that admits a diagonal form

$$(3.1) \quad Q = \langle 1, a, b, abd \rangle,$$

i.e., is given by the equation

$$s^2 + at^2 + bu^2 + abdv^2 = 0$$

where  $a, b, d \in K^\times$  and  $(s, t, u, v)$  are homogeneous coordinates in  $\mathbb{P}^3$ . Note that since  $k := \mathbb{C} \subset K$ , this form is equivalent to the form  $\langle 1, -a, -b, abd \rangle$ .

Theorem 3.17 in [Pir16] gives a general formula for the unramified  $H^2$  of the field  $K(Q)$ , in terms of the divisor of rational functions  $a, b, d \in K^\times$ , under the assumption that  $d$  is not a square.

In Section 4 we will analyze the following special case:

**Example 8.** Consider the fourfold  $X \subset \mathbb{P}^2 \times \mathbb{P}^3$  given by

$$(3.2) \quad yzs^2 + xzt^2 + xyu^2 + F(x, y, z)v^2 = 0,$$

where

$$F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz).$$

Dehomogenize by setting  $z = 1$  to obtain a quadric surface over  $k(\mathbb{P}^2)$ :

$$ys^2 + xt^2 + xyu^2 + F(x, y, 1)v^2 = 0.$$

Multiplying through by  $xy$  and absorbing squares into the variables yields

$$xS^2 + yT^2 + U^2 + xyF(x, y, 1)V^2 = 0,$$

which is of the form (3.1).

We compute the divisor  $D \subset \mathbb{P}^2$  parametrizing singular fibers of  $\pi : X \rightarrow \mathbb{P}^2$ . This is reducible, consisting of the coordinate lines (with multiplicity two) and a conic tangent to each of the lines:

$$D = \{x^2y^2z^2(x^2 + y^2 + z^2 - 2(xy + xz + yz)) = 0\}.$$

**3.3. Families containing the example.** We put the example above in a more general context. We are interested in quadric surface bundles over  $\mathbb{P}^2$  degenerating over octic plane curves  $D' \subset \mathbb{P}^2$ . The corresponding double covers  $T \rightarrow \mathbb{P}^2$  are *Horikawa* surfaces, i.e., minimal surfaces of general type such that  $c_1^2 = 2\chi - 6$ . Their moduli space is irreducible. The paper [Per82] gives examples of such surfaces with maximal Picard numbers, obtained by degenerating  $D'$ .

**Proposition 9.** *For a generic octic curve  $D' \subset \mathbb{P}^2$  there exists a smooth curve  $B$  and a flat family*

$$\phi : \mathcal{X} \rightarrow B$$

*of hypersurfaces in  $\mathbb{P}^2 \times \mathbb{P}^3$  of bidegree  $(2, 2)$  and points  $b, b' \in B$  such that*

- $X = \phi^{-1}(b)$  is the quadric bundle from Example 8 and
- $\pi' : X' = \phi^{-1}(b') \rightarrow \mathbb{P}^2$  is smooth and degenerates exactly over  $D'$ .

*Proof.* Essentially, we can smooth hypersurfaces in products of projective spaces freely.

In detail, the parameter space of  $(2, 2)$  hypersurfaces in  $\mathbb{P}^2 \times \mathbb{P}^3$  is a projective space of dimension 59; the moduli space  $\mathcal{N}_{(2,2)}$ , obtained by taking a quotient for the natural action of  $\mathrm{PGL}_3 \times \mathrm{PGL}_4$ , has dimension 36. The parameter space of octic plane curves is a projective space of dimension 44; the moduli space  $\mathcal{N}_8$  has dimension 36.

Taking determinants gives a rational map

$$\mathbb{P}^{59} \dashrightarrow \mathbb{P}^{44}$$

which descends to

$$\varphi : \mathcal{N}_{(2,2)} \dashrightarrow \mathcal{N}_8.$$

This is known to be dominant [Bea00, Proposition 4.6]; indeed, if  $D'$  is generic then the fiber  $\varphi^{-1}([D'])$  corresponds to the non-trivial two-torsion of the Jacobian of  $D'$ . Furthermore, each  $X' \in \varphi^{-1}([D'])$  is smooth.

Clearly, there is a curve  $\ell \subset \mathbb{P}^{44}$  joining  $[D']$  to  $[D]$ . Note that  $\varphi([X]) = [D]$ ; the morphism  $\varphi$  is clearly defined at  $[X]$ . Let  $B$  be the

normalization of an irreducible component of  $\mathcal{N}_{(2,2)} \times_{\mathcal{N}_8} \ell$  containing  $[X]$ . Its image in  $\mathcal{N}_8$  contains  $[D']$  by construction.  $\square$

**Remark 10.** Hypersurfaces of bidegree  $(2, 2)$  in  $\mathbb{P}^2 \times \mathbb{P}^3$  may also be regarded as conic bundles over the second factor. The degeneracy surface in  $\mathbb{P}^3$  has degree six and at least eight nodes, corresponding to rank-one fibers. As a byproduct of the proof of Theorem 1, we obtain failure of stable rationality for very general conic bundles of this type.

#### 4. THE BRAUER GROUP OF THE SPECIAL FIBER

Let  $K$  be a field. We write

$$H^n(K) = H^n(K, \mathbb{Z}/2)$$

for its  $n$ -th Galois cohomology with constant coefficients  $\mathbb{Z}/2$ . Let  $K = k(X)$  be the function field of an algebraic variety  $X$  over  $k = \mathbb{C}$ , and let  $\nu$  be a divisorial (rank-one) valuation of  $K$ . For  $n \geq 1$ , we have a natural homomorphism

$$\partial_\nu^n : H^n(K) \rightarrow H^{n-1}(\kappa(\nu)),$$

where  $\kappa(\nu)$  is the residue field of  $\nu$ . The group

$$H_{nr}^n(K) := \bigcap_\nu \text{Ker}(\partial_\nu^n)$$

is called the  $n$ -th unramified cohomology of  $K$ . It is a stable birational invariant, by definition; it vanishes if  $X$  is stably rational. Recall that for smooth projective  $X$  we have

$$\text{Br}(X)[2] = H_{nr}^2(k(X)).$$

The following proposition is similar to the examples in [Pir16, Section 3.5].

**Proposition 11.** *Let  $K = k(x, y) = k(\mathbb{P}^2)$ ,  $X \rightarrow \mathbb{P}^2$  the quadric surface bundle defined in Example 8,*

$$\alpha = (x, y) \in \text{Br}(K)[2],$$

*and  $\alpha'$  its image in  $H^2(k(X))$ . Then  $\alpha'$  is contained in  $H_{nr}^2(k(X))$  and is nontrivial; in particular,*

$$H_{nr}^2(k(X)) \neq 0.$$

*Proof.* Let  $Q$  be the generic fiber of the natural projection  $\pi : X \rightarrow \mathbb{P}^2$ . Since the discriminant of  $Q$  is not a square, the homomorphism

$$H^2(K) \rightarrow H^2(K(Q))$$

is injective [Ara75]. Thus we have to show that for every divisorial (rank-one) valuation  $\nu$  on  $K(Q)$  we have  $\partial_\nu(\alpha) = 0$ . (For simplicity, we write  $\partial_\nu$  for  $\partial_\nu^2$ .) We use standard coordinates  $x$  and  $y$  (resp.  $y$  and  $z$ , resp.  $x$  and  $z$ ) for the open charts of the projective plane. Let us first investigate the ramification of  $\alpha$  on  $\mathbb{P}^2$ ; from the definition, we only have the following nontrivial residues:

- $\partial_x(\alpha) = y$  at the line  $L_x : x = 0$ , where we write  $y$  for its class in the residue field  $k(y)$  modulo squares;
- $\partial_y(\alpha) = x$  at the line  $L_y : y = 0$ ,
- $\partial_z(\alpha) = \partial_z(z, zy) = y$  at the line  $L_z : z = 0$ , in coordinates  $y$  and  $z$  on  $\mathbb{P}^2$ .

Let  $\mathfrak{o}_\nu$  be the valuation ring of  $\nu$  in  $K(Q)$  and consider the center of  $\nu$  in  $\mathbb{P}^2$ . If  $\mathfrak{o}_\nu \supset K$  then the  $\partial_\nu(\alpha) = 0$ ; hence there are two cases to consider:

- The center is the generic point of a curve  $C_\nu$ ; we denote the corresponding residue map  $\partial_{C_\nu} : H^2(K) \rightarrow H^1(\kappa(C_\nu))$ .
- The center is a closed point  $\mathfrak{p}_\nu$ .

*Codimension 1.* The inclusion of discrete valuation rings  $\mathfrak{o}_{\mathbb{P}^2, C_\nu} \subset \mathfrak{o}_\nu$  induces a commutative diagram

$$(4.1) \quad \begin{array}{ccc} H^2(K(Q)) & \xrightarrow{\partial_\nu} & H^1(\kappa(\nu)) \\ \uparrow & & \uparrow \\ H^2(K) & \xrightarrow{\partial_{C_\nu}} & H^1(\kappa(C_\nu)) \end{array}$$

Hence we have the following cases:

- (1)  $C_\nu$  is different from  $L_x, L_y$ , or  $L_z$ . Then  $\partial_{C_\nu}(\alpha) = 0$ , so that  $\partial_\nu(\alpha')$  is zero from the diagram above.
- (2)  $C_\nu$  is one of the lines  $L_x, L_y$  or  $L_z$ . Note that modulo the equation of  $C_\nu$ , the element  $d := F(x, y, z)$  is a nonzero square, so that [Pir16, Cor. 3.12] gives  $\partial_\nu(\alpha') = 0$ .

We deduce that for any valuation  $\nu$  of  $K(Q)$  with center a codimension 1 point in  $\mathbb{P}_{\mathbb{C}}^2$  the residue  $\partial_\nu(\alpha')$  vanishes.

*Codimension 2.* Let  $\mathfrak{p}_\nu$  be the center of  $\nu$  on  $\mathbb{P}^2$ . We have an inclusion of local rings  $\mathfrak{o}_{\mathbb{P}^2, \mathfrak{p}_\nu} \subset \mathfrak{o}_\nu$  inducing the inclusion of corresponding completions  $\widehat{\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}_\nu}} \subset \widehat{\mathfrak{o}_\nu}$  with quotient fields  $K_{\mathfrak{p}_\nu} \subset K(Q)_\nu$  respectively. We have three possibilities:

- (1) If  $\mathfrak{p}_\nu \notin L_x \cup L_y \cup L_z$ , then  $\alpha$  is a cup product of units in  $\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}_\nu}$ , hence units in  $\mathfrak{o}_\nu$ , so that  $\partial_\nu(\alpha') = 0$ .
- (2) If  $\mathfrak{p}_\nu$  lies on one curve, e.g.,  $\mathfrak{p}_\nu \in L_x \setminus (\mathfrak{p}_y \sqcup \mathfrak{p}_z)$ , where  $\mathfrak{p}_y = (0, 1, 0)$  and  $\mathfrak{p}_z = (0, 0, 1)$ , then the image of  $y$  in  $\kappa(\mathfrak{p}_\nu)$  is a nonzero complex number, hence a square in  $\widehat{\mathcal{O}_{\mathbb{P}^2, \mathfrak{p}_\nu}}$ , and  $y$  is also a square in  $\widehat{\mathfrak{o}_\nu}$ . Thus  $\alpha' = 0$  in  $H^2(K(Q)_\nu, \mathbb{Z}/2)$ , and  $\partial_\nu(\alpha') = 0$ .
- (3) If  $\mathfrak{p}_\nu$  lies on two curves, e.g.,  $\mathfrak{p}_\nu = L_x \cap L_y$ , then the image of  $F(x, y, 1)$  in  $\kappa(\mathfrak{p}_\nu)$  is a nonzero complex number, hence a square. By [Pir16, Corollary 3.12], we have  $\partial_\nu(\alpha') = 0$ .

□

## 5. SINGULARITIES OF THE SPECIAL FIBER

In this section we analyze the singularities of the fourfold introduced in Section 4. Our main result is:

**Proposition 12.** *The fourfold  $X \subset \mathbb{P}^2 \times \mathbb{P}^3$ , with coordinates  $(x, y, z)$  and  $(s, t, u, v)$ , respectively, given by*

$$(5.1) \quad yzs^2 + xzt^2 + xyu^2 + F(x, y, z)v^2 = 0,$$

with

$$(5.2) \quad F(x, y, z) = x^2 + y^2 + z^2 - 2(xy + xz + yz),$$

admits a  $\text{CH}_0$ -trivial resolution of singularities.

We proceed as follows:

- identify the singular locus of  $X$ ;
- construct a resolution of singularities  $\beta : \tilde{X} \rightarrow X$ ;
- verify universal  $\text{CH}_0$ -triviality of  $\beta$ .

**5.1. The singular locus.** Here we describe the singularities explicitly using affine charts on  $\mathbb{P}^2 \times \mathbb{P}^3$ . The equations (5.1) and (5.2) are symmetric with respect to compatible permutations of  $\{x, y, z\}$  and  $\{s, t, u\}$ . In addition, there is the symmetry

$$(s, t, u, v) \rightarrow (\pm s, \pm t, \pm u, v)$$

so altogether we have an action by the semidirect product  $(\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_3$ .

*Analysis in local charts.* Let  $L_x, L_y, L_z \subset \mathbb{P}^2$  be the coordinate lines given by

$$x = 0, \quad y = 0, \quad z = 0,$$

respectively, and

$$\mathfrak{p}_x := (1, 0, 0), \quad \mathfrak{p}_y := (0, 1, 0), \quad \mathfrak{p}_z := (0, 0, 1)$$

their intersections.

The quadrics in the family (5.1) drop rank over coordinate lines  $L_x, L_y, L_z$  and over the conic  $C \subset \mathbb{P}^2$ , with equation (5.2)

$$F(x, y, z) = 0.$$

This conic is tangent to the coordinate lines in the points

$$\mathfrak{r}_x := (0, 1, 1), \quad \mathfrak{r}_y := (1, 0, 1), \quad \mathfrak{r}_z := (1, 1, 0),$$

respectively.

By symmetry, it suffices to consider just two affine charts:

**Chart 1:**  $z = u = 1$ . Equation (5.1) takes the form

$$(5.3) \quad ys^2 + xt^2 + xy + F(x, y, 1)v^2 = 0.$$

Derivatives with respect to  $s, t, v$  give

$$(5.4) \quad ys = 0, \quad xt = 0, \quad vF(x, y, z) = 0.$$

Hence  $xy = 0$ , from (5.3). Derivatives with respect to  $y, x$  give

$$(5.5) \quad s^2 + x + (2y - 2x - 2)v^2 = 0, \quad t^2 + y + (2x - 2y - 2)v^2 = 0.$$

Since  $xy = 0$ , we have two cases, modulo symmetries:

*Case 1:*  $y = 0$ ;

*Case 2:*  $x = 0, y \neq 0$ .

We analyze each of these cases:

*Case 1:*  $y = 0$ . Then  $vF(x, y, z) = 0$  (from (5.4)) implies

$$v(x - 1)^2 = 0.$$

So either  $v = 0$  or  $x = 1$ . If  $v = 0$ , from (5.5) we obtain  $s^2 + x = t = 0$ . Hence we obtain the following equations for the singular locus:

$$(5.6) \quad y = v = t = s^2 + x = 0.$$

If  $x = 1$  then (5.4) implies  $t = 0$ , and the remaining equation from (5.5) gives  $s^2 + 1 - 4v^2 = 0$ . Hence we obtain the following equations:

$$(5.7) \quad x - 1 = y = t = s^2 + 1 - 4v^2 = 0.$$

*Case 2:*  $x = 0, y \neq 0$ . From (5.4) the condition  $ys = 0$  implies  $s = 0$ . There are two more cases:  $v = 0$  or  $v \neq 0$ . If  $v = 0$  the remaining equation (5.5) gives  $t^2 + y = 0$ . Hence we obtain equations for the singularity:

$$(5.8) \quad x = v = s = t^2 + y = 0.$$

If  $v \neq 0$ , then  $F(0, y, 1) = (y - 1)^2 = 0$  from (5.4), hence  $y = 1$ . The remaining equation from (5.5) gives

$$t^2 + y + (2x - 2y - 2)v^2 = t^2 + 1 - 4v^2 = 0.$$

So we obtain equations for the singular locus:

$$(5.9) \quad x = y - 1 = s = t^2 + 1 - 4v^2 = 0.$$

**Chart 2:**  $z = s = 1$ . The equation of the quadric bundle is

$$y + xt^2 + xyu^2 + F(x, y, 1)v^2 = 0.$$

As above, derivatives with respect to  $t, v, u$  give

$$(5.10) \quad xt = 0, \quad xyu = 0, \quad vF = 0.$$

Thus  $y = 0$  from the equation. The conditions above and derivatives with respect to  $x$  and  $y$  yield

$$(5.11) \quad xt = v(x - 1)^2 = 1 + xu^2 + (-2x - 2)v^2 = t^2 + (2x - 2)v^2 = 0.$$

The second equation implies that either  $x = 1$  or  $v = 0$ .

If  $x = 1$ , we obtain:

$$(5.12) \quad x - 1 = y = t = 1 + u^2 - 4v^2 = 0.$$

If  $v = 0$ , we obtain:

$$(5.13) \quad y = t = v = 1 + xu^2 = 0.$$

Collecting these computations, we obtain the following singularities:

(1) In the chart  $ys^2 + xt^2 + xy + F(x, y, 1)v^2 = 0$  :

$$C_{y,s}^\circ : y = v = t = s^2 + x = 0$$

$$R_{y,s}^\circ : x - 1 = y = t = s^2 + 1 - 4v^2 = 0$$

$$C_{x,t}^\circ : x = v = s = t^2 + y = 0$$

$$R_{x,t}^\circ : x = y - 1 = s = t^2 + 1 - 4v^2 = 0$$

(2) In the chart  $y + xt^2 + xyu^2 + F(x, y, 1)v^2 = 0$  :

$$R_{y,u}^\circ : x - 1 = y = t = 1 + u^2 - 4v^2 = 0$$

$$C_{y,u}^\circ : y = t = v = 1 + xu^2 = 0$$

*Enumeration of strata.* The singular locus of  $X$  is a union of 6 conics. We distinguish between

- Horizontal conics  $C_x, C_y, C_z \subset X$ : these project onto the coordinate lines  $L_x, L_y, L_z \subset \mathbb{P}^2$ . We express them using our standard coordinates on  $\mathbb{P}^2 \times \mathbb{P}^3$ :

$$C_y = \{y = t = v = 0, zs^2 + xu^2 = 0\}$$

$$C_x = \{x = s = v = 0, zt^2 + yu^2 = 0\}$$

$$C_z = \{z = u = v = 0, xt^2 + ys^2 = 0\}$$

The conics intersect transversally over  $\mathfrak{p}_z, \mathfrak{p}_x, \mathfrak{p}_y \in \mathbb{P}^2$ , respectively:

$$C_x \cap C_y = \mathfrak{q}_z := (0, 0, 1) \times (0, 0, 1, 0), \quad \pi(\mathfrak{q}_z) = \mathfrak{p}_z$$

$$C_y \cap C_z = \mathfrak{q}_x := (1, 0, 0) \times (1, 0, 0, 0), \quad \pi(\mathfrak{q}_x) = \mathfrak{p}_x$$

$$C_x \cap C_z = \mathfrak{q}_y := (0, 1, 0) \times (0, 1, 0, 0), \quad \pi(\mathfrak{q}_y) = \mathfrak{p}_y$$

- Vertical conics  $R_y, R_x, R_z \subset X$ : these project to the points  $\mathfrak{r}_y, \mathfrak{r}_x, \mathfrak{r}_z \in \mathbb{P}^2$ :

$$R_y = \{x - z = y = t = 0, s^2 + u^2 - 4v^2 = 0\}$$

$$R_x = \{y - z = x = s = 0, t^2 + u^2 - 4v^2 = 0\}$$

$$R_z = \{x - y = z = u = 0, s^2 + t^2 - 4v^2 = 0\}$$

Vertical conics intersect the corresponding horizontal conics transversally in two points:

$$R_y \cap C_y = \{\mathfrak{r}_{y+}, \mathfrak{r}_{y-}\} = (1, 0, 1) \times (\pm i, 0, 1, 0)$$

$$R_x \cap C_x = \{\mathfrak{r}_{x+}, \mathfrak{r}_{x-}\} = (0, 1, 1) \times (0, 1, \pm i, 0)$$

$$R_z \cap C_z = \{\mathfrak{r}_{z+}, \mathfrak{r}_{z-}\} = (1, 1, 0) \times (1, \pm i, 0, 0)$$

*Local étale description of the singularities.* The structural properties of the resolution become clearer after identifying étale normal forms for the singularities.

We now provide a local-étale description of the neighborhood of  $\mathfrak{q}_z$ . Equation (5.3) takes the form

$$ys^2 + xt^2 + xy + F(x, y, 1)v^2 = 0.$$

At  $\mathfrak{q}_z$  we have  $F(x, y, 1) \neq 0$ , so we can set

$$v_1 = \sqrt{F(x, y, 1)}v$$

to obtain

$$ys^2 + xt^2 + xy + v_1^2 = 0.$$

Set  $x = m - n$  and  $y = m + n$  to get

$$(m + n)s^2 + (m - n)t^2 + m^2 - n^2 + v_1^2 = 0$$

or

$$m(s^2 + t^2) + n(s^2 - t^2) + m^2 - n^2 + v_1^2 = 0.$$

Then let

$$m = m_1 - (s^2 + t^2)/2 \quad \text{and} \quad n = n_1 + (s^2 - t^2)/2$$

to obtain

$$(5.14) \quad m_1^2 - n_1^2 + v_1^2 = ((s^2 + t^2)^2 - (s^2 - t^2)^2)/4 = s^2 t^2.$$

We do a similar analysis in an étale-local neighborhood of either of the points  $\mathfrak{r}_{y\pm}$ . The singular strata for  $C_y$  and  $R_y$  are given in (5.6) and (5.7):

$$\{y = t = v = s^2 + x = 0\}, \{y = t = x - 1 = s^2 + 1 - 4v^2 = 0\}.$$

We first introduce a new coordinate  $w = x - 1$ . Thus the singular stratum is the intersection of the monomial equations  $y = t = vw = 0$  and the hypersurface

$$s^2 + w + 1 - 4v^2.$$

We regard this as a local coordinate near  $\mathfrak{r}_{y\pm}$ . Equation (5.3) transforms to

$$ys^2 + wt^2 + t^2 + wy + y + v^2(-4y + (w - y)^2) = 0.$$

Regroup terms to obtain

$$y(s^2 + w + 1 - 4v^2) + t^2(1 + w) = -v^2(w - y)^2.$$

Let  $t_1 = t\sqrt{1 + w}$ ,  $s_1 = s^2 + w + 1 - 4v^2$ , and  $w_1 = w - y$  we obtain

$$(5.15) \quad ys_1 + t_1^2 = -v^2 w_1^2.$$

The normal forms (5.14) and (5.15) are both equivalent to

$$a_1^2 + a_2^2 + a_3^2 = (b_1 b_2)^2,$$

with ordinary threefold double points along the lines

$$\ell_1 = \{a_1 = a_2 = a_3 = b_1 = 0\}, \quad \ell_2 = \{a_1 = a_2 = a_3 = b_2 = 0\}.$$

A direct computation shows this is resolved by blowing up  $\ell_1$  and  $\ell_2$  in *either* order. The exceptional fibers over the generic points of  $\ell_1$  and  $\ell_2$  are smooth quadric surfaces, isomorphic to the Hirzebruch surface  $\mathbb{F}_0$ . Over the origin, we obtain a copy

$$\mathbb{F}_0 \cup_{\Sigma} \mathbb{F}_2$$

where  $\Sigma \simeq \mathbb{P}^1$  is the  $(-2)$ -curve on  $\mathbb{F}_2$  and has self-intersection 2 on  $\mathbb{F}_0$ .

By symmetry, this analysis is valid at *all* nine special points

$$\mathfrak{q}_x, \mathfrak{q}_y, \mathfrak{q}_z, \mathfrak{r}_{x\pm}, \mathfrak{r}_{y\pm}, \mathfrak{r}_{z\pm}$$

where components of the singular locus (the horizontal and vertical conics) intersect. This explains why we can blow these conics up in any order.

## 5.2. Resolution of singularities.

*What we need to compute.* We propose blowing up as follows:

- (1) blow up  $C_y$ ;
- (2) blow up the proper transform of  $C_x$ ;
- (3) blow up the proper transform of  $C_z$ ;
- (4) blow up the union of  $R_x, R_y$ , and  $R_z$ , which are disjoint.

Taking into account the symmetry, after the first step we must understand:

- What are the singularities along the proper transform of  $C_x$ ?
- What are the singularities along the proper transform of  $R_y$ ?

Of course, answering the first questions clarifies the behavior along the proper transform of  $C_z$ . And  $R_x$  and  $R_z$  behave the exactly the same as  $R_y$ .

Let  $X_1$  denote the blow up of  $C_y$  and  $E_{1,y}$  the resulting exceptional divisor. We shall see that

- $X_1$  is smooth at any point of the exceptional divisor  $E_{1,y}$ , except where  $E_{1,y}$  meets the proper transforms of  $C_x, C_z, R_y$ .
- $E_{1,y}$  is also smooth, except where it meets the proper transforms of  $C_x, C_z, R_y$ .
- The fibers of  $E_{1,y} \rightarrow C_y$  are smooth quadric surfaces away from  $\mathfrak{q}_x, \mathfrak{q}_z, \mathfrak{r}_{y\pm}$ , over which the fibers are quadric cones.

Incidentally,  $E_{1,y} \rightarrow C_y$  admits sections and  $E_{1,y}$  is rational.

*First blow up—local charts.* We describe the blow up of  $C_y$  in charts. We start in Chart 1, where  $z = u = 1$ . Local equations for the center are given in (5.6) and we have a local chart for each defining equation.

- **Chart associated with  $y$ :** Equations for the blow up of the ambient space take the form

$$v = yv_1, \quad t = yt_1, \quad s^2 + x = yw_1.$$

The equation of the proper transform of the quadric bundle is

$$w_1 + xt_1^2 + F(x, y, 1)v_1^2 = 0, \quad s^2 + x = yw_1.$$

The exceptional divisor  $E_{1,y}$  is given by  $y = 0$ , i.e.,

$$w_1 + xt_1^2 + (x - 1)^2v_1^2 = 0, \quad s^2 + x = 0.$$

The blow up is smooth at any point of the exceptional divisor in this chart. (The proper transforms of  $R_y$  and  $C_x$  do not appear in this chart.) We analyze  $E_{1,y} \rightarrow C_y$ : for any field  $\kappa/\mathbb{C}$  and  $a \in \kappa$ , the fiber above  $s = a$ ,  $x = -a^2$ ,  $y = v = t = 0$ , is given by

$$(5.16) \quad w_1 - a^2t_1^2 + (1 + a^2)^2v_1^2 = 0,$$

which is smooth in this chart. Equation (5.16) makes clear that the exceptional divisor is rational and admits a section over the center.

- **Chart associated with  $s^2 + x$ :** Equations for the blow up of the ambient space take the form

$$y = (s^2 + x)y_1, \quad v = (s^2 + x)v_1, \quad t = (s^2 + x)t_1.$$

The proper transform of the quadric bundle has equation

$$y_1 + xt_1^2 + F(x, (s^2 + x)y_1, 1)v_1^2 = 0.$$

The exceptional divisor  $E_{1,y}$  satisfies

$$y_1 + xt_1^2 + (x - 1)^2v_1^2 = 0, \quad s^2 + x = 0.$$

The blow up is smooth at any point of the exceptional divisor in this chart, as the derivative with respect to  $y_1$  is 1 and the derivative of the second equation with respect to  $y_1$  ( resp.  $x$ ) is 0 ( resp. 1). (Again, the proper transforms of  $R_y$  and  $C_x$  do not appear in this chart.) The fiber above  $s = a$ ,  $x = -a^2$ ,  $y = v = t = 0$ , is given by

$$(5.17) \quad y_1 - a^2t_1^2 + (1 + a^2)^2v_1^2 = 0,$$

which is smooth and rational in this chart.

- **Chart associated with  $t$ :** Equations for the blow up of the ambient space are

$$y = ty_1, v = tv_1, s^2 + x = tw_1$$

and the proper transform of the quadric bundle satisfies

$$y_1w_1 + x + F(x, ty_1, 1)v_1^2 = 0, s^2 + x = tw_1.$$

The exceptional divisor is given by  $t = 0$ , i.e.

$$y_1w_1 + x + (x - 1)^2v_1^2 = 0, s^2 + x = 0.$$

The blow up is smooth along the exceptional divisor, except at the point

$$t = v_1 = y_1 = s = w_1 = x = 0,$$

which lies over the point  $\mathfrak{q}_z$ . Thus the only singularity is along the proper transform of  $C_x$ . The fiber above  $s = a$ ,  $x = -a^2$ ,  $y = v = t = 0$ , is given by

$$(5.18) \quad y_1w_1 - a^2 + (1 + a^2)^2v^2 = 0,$$

which is smooth in this chart unless  $a = 0$ .

- **Chart associated with  $v$ :** The equations are

$$y = vy_1, t = vt_1, s^2 + x = vw_1$$

and

$$y_1w_1 + xt_1^2 + F(x, vy_1, 1) = 0, s^2 + x = vw_1.$$

The exceptional divisor is given by  $v = 0$ , i.e.

$$y_1w_1 + xt_1^2 + (x - 1)^2 = 0, s^2 + x = 0.$$

The blow up is smooth at any point of the exceptional divisor except for

$$y_1 = v = w_1 = t_1 = 0, x = 1, s = \pm i.$$

Thus the only singularities are along the proper transform of  $\mathfrak{r}_y$ . The fiber above  $s = a$ ,  $x = -a^2$ ,  $y = v = t = 0$ , is given by

$$(5.19) \quad y_1w_1 - a^2t_1^2 + (1 + a^2)^2 = 0,$$

which is smooth in this chart unless  $a = \pm i$ .

*Singularities above  $\mathfrak{p}_z$ .* Our goal is to show explicitly that the singularity of the blow up in the exceptional divisor  $E_{1,y}$  over  $(x, y, z) = (0, 0, 1) = \mathfrak{p}_z$  is resolved on blowing up the proper transform of  $C_x$ . It suffices to examine the chart associated with  $t$ , where we have equation

$$y_1 w_1 + x + F(x, ty_1, 1)v_1^2 = 0, \quad s^2 + x = tw_1,$$

i.e.,

$$(5.20) \quad (y_1 + t)w_1 - s^2 + F(-s^2 + tw_1, ty_1, 1)v_1^2 = 0, \quad s^2 + x = tw_1,$$

and the proper transform of  $C_x$  satisfies

$$y_1 + t = 0, \quad w_1 = s = v_1 = 0.$$

If we compute the singular locus for the equation (5.20) above, at the points of the exceptional divisor  $t = 0$  and above  $x = 0$ , we recover the equations for the proper transform of  $C_x$  in this chart.

We analyze  $X_2$ , the blowup along the proper transform of  $C_x$ . In any chart above  $y_1 = t = 0$  we have  $F = 1$  so étale locally we can introduce a new variable  $v_2 = \sqrt{F}v_1$  to obtain

$$(y_1 + t)w_1 - s^2 + v_2^2 = 0.$$

After the change of variables  $y_2 = y_1 + t$ :

$$y_2 w_1 - s^2 + v_2^2 = 0,$$

the singular locus is  $y_2 = s = w_1 = v_2 = 0$ . Here  $t$  is a free variable corresponding to an  $\mathbb{A}^1$ -factor. This is the product of an ordinary threefold double point with curve, thus is resolved on blowing up the singular locus. Note the exceptional divisor is a smooth quadric surface bundle over the proper transform of  $C_x$ , over this chart. (There is a singular fiber over the point where it meets the proper transform of  $C_z$ .)

*Singularities above  $\mathfrak{r}_y = (1, 0, 1) \in \mathbb{P}^2$ .* By the analysis above, we have only to consider the chart of the first blowup associated with  $v$ . Recall that it is obtained by setting

$$y = vy_1, \quad t = vt_1, \quad s^2 + x = vw_1$$

with equation

$$y_1 w_1 + xt_1^2 + F(x, vy_1, 1) = 0.$$

The exceptional divisor is given by  $v = 0$ . The proper transform  $R'_y$  of the conic

$$R_y : x - 1 = y = t = 0, \quad s^2 + 1 - 4v^2 = 0$$

is then

$$(5.21) \quad x - 1 = y_1 = t_1 = 0, \quad w_1 - 4v = 0, \quad s^2 + 1 - 4v^2 = 0.$$

We obtain these equations by inverting the local equation for the exceptional divisor. Eliminating  $x$  from the equation of  $X_1$  yields an equation that can be put in the form

$$y_1(w_1 - 4v) + (-s^2 + vw_1)t_1^2 + (s^2 - vw_1 + vy_1 + 1)^2 = 0.$$

Writing  $w_2 = w_1 - 4v$  we obtain

$$y_1w_2 + (-s^2 + vw_2 + 4v^2)t_1^2 + (s^2 - vw_2 - 4v^2 + vy_1 + 1)^2 = 0.$$

The curve  $R'_y$  may be expressed as a complete intersection

$$y_1 = w_2 = t_1 = 0, \quad \sigma := (s^2 - 4v^2 + 1) + v(y_1 - w_2) = 0;$$

the coefficient

$$c := -s^2 + vw_2 + 4v^2$$

is non-vanishing along  $R'_y$  in this chart so we may introduce an étale local coordinate  $t_2 = \sqrt{ct_1}$ . Then our equation takes the form

$$y_1w_2 + t_2^2 + \sigma^2 = 0.$$

In other words, we have ordinary threefold double points along each point of  $R'_y$ . Blowing up  $R'_y$  resolves the singularity, and the exceptional divisor over  $R'_y$  is fibered in smooth quadric surfaces.

**5.3.  $\text{CH}_0$ -triviality of the resolution.** Let  $E_{1,y}$  denote the exceptional divisor after blowing up  $C_y$ . We've seen that the projection  $E_{1,y} \rightarrow C_y$  is a quadric surface bundle. The fibers are smooth away from  $\mathfrak{q}_x, \mathfrak{q}_z$ , and  $\mathfrak{r}_{y\pm}$ ; over these points the fibers are quadric cones.

Let  $E_{1,x}$  denote the exceptional divisor after blowing up the proper transform  $C'_x$  of  $C_x$ . The fibration  $E_{1,x} \rightarrow C'_x$  is also a quadric surface bundle. The fibers are smooth away from  $\mathfrak{q}_y$  and  $\mathfrak{r}_{x\pm}$ , where the fibers are quadric cones.

Let  $E_{1,z}$  denote the exceptional divisor on blowing up the proper transform  $C'_z$  of  $C_z$ , after the first two blow ups. Again  $E_{1,z} \rightarrow C'_z$  is a quadric surface bundle, smooth away from  $\mathfrak{r}_{z\pm}$ ; the fibers over these points are quadric cones.

Finally, we blow up the proper transforms  $R'_x, R'_y, R'_z$  of the disjoint vertical conics. The local computations above show that the resulting fourfold  $\tilde{X}$  is smooth and the exceptional divisors

$$E_{2,x} \rightarrow R'_x, \quad E_{2,y} \rightarrow R'_y, \quad E_{2,z} \rightarrow R'_z,$$

are smooth rational quadric surface bundles.

To summarize, fibers of  $\beta : \tilde{X} \rightarrow X$  are one of the following:

- if  $x$  is not contained in any of conics,  $\beta^{-1}(x)$  is a point;
- if  $x$  is contained in exactly one of the conics,  $\beta^{-1}(x)$  is a smooth quadric surface isomorphic to  $\mathbb{F}_0$ ; when  $x$  is a generic point of one of the conics, then  $\beta^{-1}(x)$  is rational over the residue field of  $x$ ;
- if  $x$  is contained in two of the conics,  $\beta^{-1}(x) = \mathbb{F}_0 \cup_{\Sigma} \mathbb{F}_2$ , where  $\mathbb{F}_2$  is the proper transform of a quadric cone appearing as a degenerate fiber,  $\Sigma \subset \mathbb{F}_2$  is the  $(-2)$  curve, and  $\Sigma \subset \mathbb{F}_0$  has self-intersection 2.

An application of Proposition 3 yields that  $\beta$  is universally  $\text{CH}_0$ -trivial.

## 6. ANALYSIS OF HODGE CLASSES

Our approach follows Section 2 of [Voi15a]. As explained in Proposition 6, a quadric surface bundle over a rational surface  $\pi : X \rightarrow S$  is rational provided  $X$  admits an integral class of type  $(2, 2)$  meeting the fibers of  $\pi$  in odd degree. Here we analyze how these classes occur.

We start by reviewing the Hodge-theoretic inputs. Let  $\mathcal{Y} \rightarrow B$  be the family of all smooth hypersurfaces in  $\mathbb{P}^2 \times \mathbb{P}^3$  of bidegree  $(2, 2)$ , i.e.,  $B$  is the complement of the discriminant in  $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(2, 2)))$ . For each  $b \in B$ , let  $Y_b$  denote the fiber over  $b$ . The Lefschetz hyperplane theorem gives Betti/Hodge numbers

- $b_{2i+1}(Y_b) = 0$
- $b_2(Y_b) = h^{1,1}(Y_b) = 2$ ,  $b_6(Y_b) = h^{3,3}(Y_b) = 2$ .

We compute  $b_4(Y_b)$  by analyzing  $Y_b \rightarrow \mathbb{P}^2$ ; its degeneracy divisor is an octic plane curve  $D_b$ , of genus 21. Indeed, the fibers away from  $D_b$  are smooth quadric surfaces and the fibers over  $D_b$  are quadric cones, so we have

$$\begin{aligned} \chi(Y_b) &= \chi(\mathbb{P}^1 \times \mathbb{P}^1) \chi(\mathbb{P}^2 \setminus D_b) + \chi(\text{quadric cone}) \chi(D_b) \\ &= 4 \cdot (3 - (-40)) + 3 \cdot (-40) = 52. \end{aligned}$$

It follows that  $b_4(Y_b) = 46$ .

We extract the remaining Hodge numbers using techniques of Griffiths and Donagi [DG84] for hypersurfaces in projective space, extended to the toric case by Batyrev and Cox. Let  $F$  be the defining equation of bidegree  $(2, 2)$  and consider the bigraded *Jacobian ring*:

$$\text{Jac}(F) = \mathbb{C}[x, y, z, s, t, u, v]/\mathcal{I}(F),$$

where  $\mathcal{I}(F)$  is the ideal of partials of  $F$ . Note the partials satisfy Euler relations:

$$(6.1) \quad x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} = 2F = s \frac{\partial F}{\partial s} + t \frac{\partial F}{\partial t} + u \frac{\partial F}{\partial u} + v \frac{\partial F}{\partial v}.$$

Consider the *vanishing cohomology*

$$H^4(Y_b)_{van} := H^4(Y_b)/H^4(\mathbb{P}^2 \times \mathbb{P}^3),$$

i.e., we quotient by  $\langle h_1^2, h_1 h_2, h_2^2 \rangle$  where  $h_1$  and  $h_2$  are pull-backs of the hyperplane classes of  $\mathbb{P}^2$  and  $\mathbb{P}^3$  respectively. Then we have [BC94, Theorem 10.13]:

- $H^{4,0}(Y_b) = H^{4,0}(Y_b)_{van} = \text{Jac}(F)_{(-1,-2)} = 0$
- $H^{3,1}(Y_b) = H^{3,1}(Y_b)_{van} \simeq \text{Jac}(F)_{(1,0)} = \mathbb{C}[x, y, z]_1 \simeq \mathbb{C}^3$
- $H^{2,2}(Y_b)_{van} \simeq \text{Jac}(F)_{(3,2)} \simeq \mathbb{C}^{37}$
- $H^{1,3}(Y_b) = H_{1,3}(Y_b)_{van} \simeq \text{Jac}(F)_{(5,4)} \simeq \mathbb{C}^3$ .

The first two dimension computations imply the others by the formula

$$b_4(Y_b) = \sum_{p+q=4} h^{p,q}(Y_b);$$

or one may compute the Hilbert function of an ideal generated by three forms of degree (1, 2) and four forms of degree (2, 1), subject to the relations (6.1) but otherwise generic.

We recall the technique of [Voi07, 5.3.4]:

**Proposition 13.** *Suppose there exists a  $b_0 \in B$  and  $\gamma \in H^{2,2}(Y_{b_0})$  such that the infinitesimal period map evaluated at  $\gamma$*

$$\bar{\nabla}(\gamma) : T_{B,b_0} \rightarrow H^{1,3}(Y_{b_0})$$

*is surjective. Then for any  $b \in B$  and any Euclidean neighborhood  $b \in B' \subset B$ , the image of the natural map (composition of inclusion with local trivialization):*

$$T_b : H^{2,2}(Y_{B'}, \mathbb{R}) \rightarrow H^4(Y_b, \mathbb{R})$$

*contains an open subset  $V_b \subset H^4(Y_b, \mathbb{R})$ .*

Note that the image is the set of real degree-four classes that are of type (2, 2) for some  $b' \in B'$ . We should point out that our variation of Hodge structures is of weight two after a suitable Tate twist.

The infinitesimal condition is easy to check here. Since

$$B \subset \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(2, 2)))$$

we may identify

$$T_{B,b_0} = (\mathbb{C}[x, y, z; s, t, u, v] / \langle F_0 \rangle)_{(2,2)},$$

where  $F_0$  is the defining equation of  $Y_{b_0}$ . The infinitesimal period map

$$T_{B,b_0} \rightarrow \mathrm{Hom}(H^{2,2}(Y_b), H^{1,3}(Y_b))$$

is given by multiplication

$$(\mathbb{C}[x, y, z; s, t, u, v] / \langle F_0 \rangle)_{(2,2)} \times \mathrm{Jac}(F_0)_{(3,2)} \rightarrow \mathrm{Jac}(F_0)_{(5,4)}.$$

For fixed  $\gamma \in \mathrm{Jac}(F_0)_{(3,2)}$ , the differential in Voisin's hypothesis is computed by multiplying  $\gamma$  with the elements of bidegree  $(2, 2)$  [Voi07, Theorem 6.13].

**Example 14.** Consider the hypersurface  $Y_{b_0} \subset \mathbb{P}^2 \times \mathbb{P}^3$  with equation

$$\begin{aligned} F_0 = & (u^2 + uv + ts)x^2 + (-t^2 + u^2 - v^2 - s^2)xy + (t^2 + uv + ts)y^2 \\ & + (-t^2 + u^2 - v^2 - s^2)xz + (t^2 - 16tu - u^2 + v^2 + s^2)yz \\ & + (-3uv - 3ts + s^2)z^2. \end{aligned}$$

We computed the Jacobian ring using Macaulay 2 [GS]. In particular, we verified that

- $\mathrm{Jac}(F_0)_{(m_1, m_2)} = 0$  for

$$(m_1, m_2) \geq (13, 2), (7, 3), (3, 5),$$

so in particular  $Y_{b_0}$  is smooth;

- the monomials  $\{xz^4v^4, yz^4v^4, z^5v^4\}$  form a basis for  $\mathrm{Jac}(F_0)_{(5,4)}$ .

Setting  $\gamma = z^3v^2$ , the multiples of  $\gamma$  generate  $\mathrm{Jac}(F_0)_{(5,4)}$ . Hence this example satisfies Voisin's hypothesis on the differential of the period map.

**Proposition 15.** *Consider the Noether-Lefschetz loci*

$$\{b \in B : Y_b \text{ admits an integral class of type } (2, 2) \text{ meeting the fibers of } Y_b \rightarrow \mathbb{P}^2 \text{ in odd degree}\}.$$

*These are dense in the Euclidean topology on  $B$ .*

*Proof.* Let  $\Lambda$  be the middle cohomology lattice of  $Y_b$ . It remains to check that classes in  $\Lambda$  with odd degree with respect to  $Y_b \rightarrow \mathbb{P}^2$  are dense in  $\Lambda \otimes \mathbb{Q}$ . There exist odd-degree classes as we can write down a bidegree  $(2, 2)$  hypersurface containing a constant section of  $\mathbb{P}^2 \times \mathbb{P}^3 \rightarrow \mathbb{P}^2$ . Adding even degree classes to this fixed class yields a dense collection of the desired classes.  $\square$

The Noether-Lefschetz loci produced by this argument have codimension at most three in moduli; each is an algebraic subvariety of  $B \subset \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^3}(2, 2))) \simeq \mathbb{P}^{59}$  [CDK95]. Any projective threefold in  $\mathbb{P}^{59}$  will meet the closures of infinitely many of these loci.

## 7. PROOF OF THEOREM 1

We assemble the various ingredients developed above:

- (1) Theorem 4 guarantees that a very general hypersurface of bidegree  $(2, 2)$  in  $\mathbb{P}^2 \times \mathbb{P}^3$  fails to be stably rational, provided we can find a special  $X$  satisfying its hypotheses.
- (2) The candidate example is introduced in Example 8.
- (3) In Section 4, we show that  $X$  has non-trivial unramified second cohomology. This verifies the first hypothesis of Theorem 4.
- (4) In Section 5, we analyze the singularities of  $X$ , checking that it admits a resolution with universally  $\text{CH}_0$ -trivial fibers.
- (5) Proposition 6 gives a cohomological sufficient condition for rationality of  $(2, 2)$  hypersurfaces in  $\mathbb{P}^2 \times \mathbb{P}^3$ ; Proposition 15 shows this condition is satisfied over a dense subset of the moduli space.

Consider a family  $\phi : \mathcal{X} \rightarrow B$  of smooth  $(2, 2)$  hypersurfaces in  $\mathbb{P}^2 \times \mathbb{P}^3$  over a connected base  $B$ . If the base meets both the locus parametrizing non-stably rational varieties and the Noether-Lefschetz loci then  $\phi$  has both rational and irrational fibers.

**Remark 16.** Concrete examples of rational hypersurfaces  $X_{rat} \subset \mathbb{P}^2 \times \mathbb{P}^3$  of bidegree  $(2, 2)$  are easy to produce, e.g., those containing a constant section of the first projection. Any very general pencil containing  $X_{rat}$  will have both rational and irrational fibers.

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