FINAL MATH-GA 2350.001 DIFFERENTIAL GEOMETRY I

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1. Let p, q be integers with $p > 0, q \geq 0$ and let E be the real vector space $\mathbb{R}^p \times \mathbb{R}^q$. We denote by $(x_1, x_2, \dots x_p, y_1, \dots y_q)$ the coordinates of E. For $x = (x_1, x_2, \dots x_p, y_1, \dots y_q)$, we define
$$Q(x) = x_1^2 + \dots + x_p^2 - y_1^2 - \dots - y_q^2.$$
Let $M \subset E$ be defined by equation
$$Q(x) = 1.$$
   (a) Show that M is a regular submanifold of E. What is the dimension of M?
   The differential of Q at a point $x = (x_1, x_2, \dots x_p, y_1, \dots y_q)$ is a map
$$(X_1, \dots X_p, Y_1, \dots Y_q) \mapsto 2\sum_{i=1}^{p} x_i X_i - 2\sum_{j=1}^{q} y_j Y_j.$$
   Hence it is non-zero for $x \neq 0$, hence submersive, as a map to $\mathbb{R}$. The image of Q contains 1 since $p \neq 0$, so that $M = Q^{-1}(1)$ for a submersive map Q on $E \setminus 0$. Hence M is regular submanifold of dimension $p+q-1$.
   (b) Assume that $p, q \geq 2$ and define, for $x = (x_1, x_2, \dots x_p, y_1, \dots y_q)$:
$$A(x) = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2},\ B(x) = y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2},\ C(x) = y_1\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial y_1}.$$
   Show that A, B and C are complete vector fields on M. Compute the flow for C. Compute $[A, C]$.
   The tangent space of M at $x = (x_1, x_2, \dots x_p, y_1, \dots y_q)$ is
$$T_xM = ker\, dQ_x = \{(X_1, \dots X_p, Y_1, \dots Y_q) \in E,\ \sum_{i=1}^{p} x_i X_i - \sum_{j=1}^{q} y_j Y_j = 0.$$
   We then see that $A(x), B(x), C(x) \in T_xM$. In addition, the maps $x \mapsto A(x), x \mapsto B(x), x \mapsto C(x)$ are smooth, so that A, B, C are smooth vector fields on M. Let $\phi_t : M \to M$ be the map
$$(x_1, x_2, \dots x_p, y_1, \dots y_q) \mapsto (x_1 cosh\, t + y_1 sinh\, t, x_2, \dots x_p, x_1 sinh\, t + y_1 cosh\, t, y_2 \dots y_q).$$
   Then $\phi_0(x) = x$ and $\frac{d}{dt}\phi_t(x) = C(\phi_t(x))$. By unicity, ϕ_t is the flow of C. Similarly we obtain the flows for A and B:
$$(x_1, x_2, \dots x_p, y_1, \dots y_q) \mapsto (x_1 cos\, t + y_1 sin\, t, -x_1 sin\, t + x_2 cos\, t, \dots x_p, y_1, y_2 \dots y_q),$$
$$(x_1, x_2, \dots x_p, y_1, \dots y_q) \mapsto (x_1, x_2, \dots x_p, y_1 cos\, t + y_2 sin\, t, -y_1 sin\, t + y_2 cos\, t, \dots y_q).$$
   These flows are defined on $\mathbb{R}$, so that A, B, C are complete. Since C does not depend on x_2 and A does not depend on y_1, we compute
$$[A, C](x) = x_2\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial x_2}.$$

(c) Show that M retracts on $M \cap (\mathbb{R}^p \times \{0\})$.
For example, the following map is the requested retraction

$$h(t,x) = \frac{1}{\sqrt{(x_1^2 + \ldots + x_p^2) - (1-t)^2(y_1^2 + \ldots y_q^2)}}(x_1, \ldots x_p, (1-t)y_1, \ldots (1-t)y_q.)$$

(d) Compute the de Rham cohomology $H^i_{dR}(M)$ for $i \geq 0$.
By the previous question, the cohomology of M coincide with cohomology of $M \cap (\mathbb{R}^p \times \{0\})$, hence the cohomology of spheres.

2. Let

$$G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} a > 0, b \in \mathbb{R}\} \subset GL_2(\mathbb{R}).$$

(a) Show that G is a Lie group.
First note that G is stable by multiplication and by taking inverses. It is then enough to verify that the map $x, y \to xy^{-1}$ is smooth ($x, y \in G$), that is straightforward. Another argument: $GL_2(\mathbb{R})$ is a Lie group, so the map $x, y \mapsto xy^{-1}$ is smooth, hence the restriction of this map to a closed subset G is also smooth.

(b) Let $\mathfrak{g}$ be the Lie algebra of G and let $A, B \in \mathfrak{g}$ be defined by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Compute $[A, B]$.
From a resilt in the lectures, since $G \subset GL_2(\mathbb{R})$ is closed, the Lie bracket on G is the restriction of the Lie bracket on $GL_2(\mathbb{R})$. Hence $[A, B] = AB - BA = B$.

(c) Describe all one-parameter subgroups of G.
Again, the exponential on G is the restriction of the exponential on $GL_2(\mathbb{R})$, hence it's the usual exponential of a matrix. Note that

$$exp \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^x & \frac{e^x - 1}{x} y \\ 0 & 1. \end{pmatrix}$$

Note that A and B generate $\mathfrak{g}$, so that for $X \in \mathfrak{g}$ one can write $X = xA + yB$. Then $exp(xA + yB) = \begin{pmatrix} e^x & \frac{e^x - 1}{x} y \\ 0 & 1. \end{pmatrix}$. The one parameter subgroups of G all of type $t \mapsto exp(tX)$ for an element $X \in G$. With the description above, we obtain that all one parameter subgroups are of type $t \mapsto \begin{pmatrix} e^{tx} & \frac{e^{tx} - 1}{x} y \\ 0 & 1. \end{pmatrix}$

3. Show that there is no Lie group structure on a sphere of dimension n with $n > 0$ even. (Hint: you can use the conclusion in the last homework.)

Assume that there is a Lie group structure on S^n *with* n *even. Then, for* $v \in T_eS^n$ *one could define the following vector field:* $X(x) = T_eL_x(v)$. *If* $v \neq 0$, *this vector field is nowhere vanishing. This is a contradiction (see HW9, ex.4).*