FINAL MATH-GA 2350.001 DIFFERENTIAL GEOMETRY I

December, 12, 2016, 1.25-3.15pm.

Authorized documents: Jeffrey M. Lee *Manifolds and Differential Geometry*, class notes from the lectures, homeworks. No other books, no internet access or electronic materials.

1. Let p, q be integers with $p > 0, q \geq 0$ and let E be the real vector space $\mathbb{R}^p \times \mathbb{R}^q$. We denote by $(x_1, x_2, \dots x_p, y_1, \dots y_q)$ the coordinates of E. For $x = (x_1, x_2, \dots x_p, y_1, \dots y_q)$, we define

$$Q(x) = x_1^2 + \dots + x_p^2 - y_1^2 - \dots - y_q^2.$$

Let $M \subset E$ be defined by equation

$$Q(x) = 1.$$

   (a) Show that M is a smooth manifold. What is the dimension of M?

   (b) Assume that $p, q \geq 2$ and define, for $x = (x_1, x_2, \dots x_p, y_1, \dots y_q)$:

$$A(x) = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2},\ B(x) = y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2},\ C(x) = y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1}.$$

   Show that A, B and C are complete vector fields on M. Compute the flow for C. Compute $[A, C]$.

   (c) Show that M retracts on $M \cap (\mathbb{R}^p \times \{0\})$.

   (d) Compute the de Rham cohomology $H^i_{dR}(M)$ for $i \geq 0$.

2. Let

$$G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} a > 0, b \in \mathbb{R}\} \subset GL_2(\mathbb{R}).$$

   (a) Show that G is a Lie group.

   (b) Let $\mathfrak{g}$ be the Lie algebra of G and let $A, B \in \mathfrak{g}$ be defined by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

   Compute $[A, B]$.

   (c) Describe all one-parameter subgroups of G.

3. Show that there is no Lie group structure on a sphere of dimension n with $n > 0$ even. (Hint: you can use that any vector field on the sphere S^{2n} has at least one zero.)