NOTE ON THE COUNTEREXAMPLES FOR THE INTEGRAL TATE CONJECTURE OVER FINITE FIELDS

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Abstract. In this note we discuss some examples of non torsion and non algebraic cohomology classes for varieties over finite fields. The approach follows the construction of Atiyah-Hirzebruch and Totaro.

1. Introduction

Let k be a finite field and let X be a smooth and projective variety over k. Denote $\bar{k}$ an algebraic closure of k and $\mathfrak{g} = Gal(\bar{k}/k)$. Let ℓ be a prime, $\ell \neq char(k)$. The Tate conjecture [19] predicts that the cycle class map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Q}_\ell \to \bigcup_U H^{2i}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Q}_\ell(i))^U,$$

where the union is over all open subgroups U of $\mathfrak{g}$, is surjective.

In the integral version one is interested in the cokernel of the cycle class map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_U H^{2i}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(i))^U. \tag{1.1}$$

This map is not surjective in general: the counterexamples of Atiyah-Hirzebruch [1], revisited by Totaro [20], to the integral version of the Hodge conjecture, provide also counterexamples to the integral Tate conjecture [3]. More precisely, one constructs an ℓ-torsion class in $H^4_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(2))$, which is not algebraic, for some smooth and projective variety X. However, one then wonders if there exists an example of a variety X over a finite field, such that the map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_U H^{2i}_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(i))^U/torsion \tag{1.2}$$

is not surjective ([12, 3]). In the context of an integral version of the Hodge conjecture, Kollár [11] constructed such examples of curve

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classes. Over a finite field, Schoen [17] has proved that the map (1.2) is always surjective for curve classes, if the Tate conjecture holds for divisors on surfaces.

In this note we follow the approach of Atiyah-Hirzebruch and Totaro and we produce examples where the map (1.2) is not surjective for $\ell = 2, 3$ or 5.

Theorem 1.1. *Let ℓ be a prime from the following list: $\ell = 2, 3$ or 5. There exists a smooth and projective variety X over a finite field k, char$k \neq \ell$, such that the cycle class map*

$$CH^2(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_U H^4_{\acute{e}t}(X_{\bar{k}}, \mathbb{Z}_\ell(2))^U/torsion$$

is not surjective.

As in the examples of Atiyah-Hirzebruch and Totaro, our counterexamples are obtained as a projective approximation of the cohomology of classifying spaces of some simple simply connected groups, having ℓ-torsion in its cohomology. The non algebraicity of a cohomology class is obtained by means of motivic cohomology operations: one establishes that the operation Q_1 does not vanish on some class of degree 4, but it always vanishes on the algebraic classes. This is done in section 2. Next, in section 3 we discuss some properties of classifying spaces in our context and finally we construct a projective variety approximating the cohomology of these spaces in small degrees in section 4.

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2. Motivic version of Atiyah-Hirzebruch arguments, revisited

2.1. **Operations.** Let k be a perfect field with $char(k) \neq \ell$ and let $\mathcal{H}.(k)$ be the motivic homotopy theory of pointed k-spaces (see [14]). For $X \in \mathcal{H}.(k)$, denote by $H^{*,*'}(X, \mathbb{Z}/\ell)$ the motivic cohomology groups with $\mathbb{Z}/\ell$-coefficients (*loc.cit.*). If X is a smooth variety over k, note that one has an isomorphism $CH^*(X)/\ell \xrightarrow{\sim} H^{2*,*}(X,\mathbb{Z}/\ell)$.

Voevodsky [22] defined the reduced power operations P^i and the Milnor's operations Q_i on $H^{*,*'}(X,\mathbb{Z}/\ell)$:

$$P^i : H^{*,*'}(X,\mathbb{Z}/\ell) \to H^{*+2i(\ell-1),*'+i(\ell-1)}(X,\mathbb{Z}/\ell), i \geq 0$$

$$Q_i : H^{*,*'}(X,\mathbb{Z}/\ell) \to H^{*+2\ell^i-1,*'+(\ell^i-1)}(X,\mathbb{Z}/\ell), i \geq 0,$$

where $Q_0 = \beta$ is the Bockstein operation of degree $(1,0)$ induced from the short exact sequence $0 \to \mathbb{Z}/\ell \xrightarrow{\times\ell} \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 0$ (see also [16]).

One of the key ingredients for this construction is the following computation of the motivic cohomology of the classifying space $B\mu_\ell$ ([22]):

Lemma 2.1. ([22, §6]) *For each object $X \in \mathcal{H}.(k)$, the graded algebra $H^{*,*'}(X \times B\mu_\ell, \mathbb{Z}/\ell)$ is generated over $H^{*,*'}(X,\mathbb{Z}/\ell)$ by*
x, $deg(x) = (1,1)$ and y, $deg(y) = (2,1)$
with $\beta(x) = y$ and $x^2 = \begin{cases} 0 & \ell \text{ is odd} \\ \tau y + \rho x & \ell = 2 \end{cases}$
where $0 \neq \tau \in H^{0,1}(Spec(k),\mathbb{Z}/\ell) \cong \mu_\ell$ and $\rho = (-1) \in k^*/(k^*)^2 \cong K_1^M(k)/2 \cong H^{1,1}(Spec(k),\mathbb{Z}/2)$.

For what follows, we assume that k contains a primitive ℓ^2-th root of unity ξ, so that $B\mathbb{Z}/\ell \xrightarrow{\sim} B\mu_\ell$ and $\beta(\tau) = \xi^\ell$ $(= \rho$ for $p = 2)$ is zero in $k^*/(k^*)^\ell = H^{1,1}_{et}(Spec(k);\mathbb{Z}/\ell)$.

We will need the following properties:

Proposition 2.2. (i) *$P^i(x) = 0$ for $i > m/2$ and $x \in H^{m,n}(X,\mathbb{Z}/\ell)$;*
(ii) *$P^i(x) = x^\ell$ for $x \in H^{2i,i}(X,\mathbb{Z}/\ell)$;*
(iii) *for X smooth the operation*

$$Q_i : CH^m(X)/\ell = H^{2m,m}(X,\mathbb{Z}/\ell) \to H^{2m+2\ell^i-1,m+(\ell^i-1)}(X,\mathbb{Z}/\ell)$$

is zero ;
(iv) *$Op.(\tau x) = \tau Op.(x)$ for $Op. = \beta, Q_i$ or P^j;*
(v) *$Q_i = [P^{\ell^{i-1}}, Q_{i-1}]$.*

Proof. See [22, §9]; for (iii) one uses that $H^{m,n}(X,\mathbb{Z}/\ell) = 0$ if $m-2n > 0$ and X is a smooth variety over k, (iv) follows from the Cartan formula for the motivic cohomology.

2.2. **Computations for $B\mathbb{Z}/\ell$.** The computations in this section are similar to [1, 20, 21].

Lemma 2.3. *In $H^{*,*'}(B\mathbb{Z}/\ell, \mathbb{Z}/\ell)$, we have $Q_i(x) = y^{\ell^i}$ and $Q_i(y) = 0$.*

Proof. By definition $Q_0(x) = \beta(x) = y$. Using induction and Proposition 2.2, we compute

$$\begin{aligned} Q_i(x) = P^{\ell^{i-1}} Q_{i-1}(x) - Q_{i-1} P^{\ell^{i-1}}(x) &= P^{\ell^{i-1}} Q_{i-1}(x) \\ &= P^{\ell^{i-1}}(y^{\ell^{i-1}}) = y^{\ell^i}. \end{aligned}$$

Then $Q_1(y) = -Q_0 P^1(y) = -\beta(y^\ell) = 0$. For $i > 1$, using induction and Proposition 2.2 again, we conclude that $Q_i(y) = -Q_{i-1} P^{\ell^{i-1}}(y) = 0$.

□

Let $G = (\mathbb{Z}/\ell)^3$. As above, we assume that k contains a primitive ℓ^2-th root of unity. From Lemma 2.1, we have an isomorphism

$$H^{*,*'}(BG, \mathbb{Z}/\ell) \cong H^{*,*'}(Spec(k), \mathbb{Z}/\ell)[y_1, y_2, y_3] \otimes \Lambda(x_1, x_2, x_3)$$

where $\Lambda(x_1, x_2, x_3)$ is isomorphic to the $\mathbb{Z}/\ell$-module generated by 1 and $x_{i_1}...x_{i_s}$ for $i_1 < ... < i_s$ and $x_i x_j = -x_j x_i$ $(i \leq j)$, with $\beta(x_i) = y_i$ and $x_i^2 = \tau y_i$ for $\ell = 2$.

Lemma 2.4. *Let $x = x_1 x_2 x_3$ in $H^{3,3}(BG, \mathbb{Z}/\ell)$. Then*

$$Q_i Q_j Q_k(x) \neq 0 \in H^{2*,*}(BG, \mathbb{Z}/\ell) \quad \text{for } i < j < k.$$

Proof. Using Proposition 2.2(v) and Cartan formula (2.2(iv)), we get

$$Q_k(x) = y_1^{\ell^k} x_2 x_3 - y_2^{\ell^k} x_1 x_3 + y_3^{\ell^k} x_1 x_2.$$

Then we deduce

$$Q_i Q_j Q_k(x) = \sum_{\sigma \in S_3} \pm y_{\sigma(1)}^{\ell^k} y_{\sigma(2)}^{\ell^j} y_{\sigma(3)}^{\ell^i} \neq 0 \in \mathbb{Z}/\ell[y_1, y_2, y_3].$$

□

3. Exceptional Lie groups

Let (G, ℓ) be a simple simply connected Lie group and a prime number from the following list:

$$(G, \ell) = \begin{cases} G_2, \ell = 2, \\ F_4, \ell = 3, \\ E_8, \ell = 5. \end{cases} \tag{3.1}$$

Then G is 2-connected and $H^3(G,\mathbb{Z}) \cong \mathbb{Z}$. Hence BG, viewed as a topological space, is 3-connected and $H^4(BG,\mathbb{Z}) \cong \mathbb{Z}$ (see [13] for example). We write $x_4(G)$ for a generator of $H^4(BG,\mathbb{Z})$.

Given a field k with $char(k) \neq \ell$, let us denote by G_k the (split) reductive algebraic group over k corresponding to the Lie group G.

The Chow ring $CH^*(BG_k)$ has been defined by Totaro [21]. More precisely, one has

$$BG_k = \varinjlim (U/G_k), \tag{3.2}$$

where $U \subset W$ is an open set in a linear representation W of G_k, such that G_k acts freely on U. One can then identify $CH^i(BG_k)$ with the group $CH^i(U/G_k)$ if $\mathrm{codim}_W(W \setminus U) > i$, the group $CH^i(BG_k)$ is then independent of a choice of such U. Similarly, one can define the étale cohomology groups $H^i_{\acute{e}t}(BG_k,\mathbb{Z}_\ell(j))$ and the motivic cohomology groups $H^{*,*'}(BG_k,\mathbb{Z}/\ell)$ (see [7]), the latter coincide with the motivic cohomology groups of [14] (cf. [7, Proposition 2.29]). We also have the cycle class map

$$cl : CH^*(BG_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_U H^{2*}_{\acute{e}t}(BG_{\bar{k}},\mathbb{Z}_\ell(*))^U, \tag{3.3}$$

where the union is over all open subgroups U of $Gal(\bar{k}/k)$.

The following proposition is known.

Proposition 3.1. *Let (G,ℓ) be a group and a prime number from the list (3.1). Then*

(i) *the group G has a maximal elementary non toral subgroup of rank 3:*
$$i : A \simeq (\mathbb{Z}/\ell)^3 \subset G;$$
(ii) *$H^4(BG,\mathbb{Z}/\ell) \simeq \mathbb{Z}/\ell$, generated by the image x_4 of the generator $x_4(G)$ of $H^4(BG,\mathbb{Z}) \simeq \mathbb{Z}$;*
(iii) *$Q_1(i^*x_4) = Q_1Q_0(x_1x_2x_3)$, in the notations of Lemma 2.4. In particular, $Q_1(i^*x_4)$ is non zero.*

Proof. For (i) see [5], for the computation of the cohomology groups with $\mathbb{Z}/\ell$-coefficients in (ii) see [13] VII 5.12; (iii) follows from [10] for $\ell = 2$ and [8, Proposition 3.2] for $\ell = 3,5$ (see [9] as well). □

4. Algebraic approximation of BG

Write

$$BG_k = \varinjlim (U/G_k) \tag{4.1}$$

as in the previous section. Using proposition 3.1 and a specialization argument, we will first construct a quasi-projective algebraic variety X

over k as a quotient $X = U/G_k$ (where $codim_W(W \setminus U)$ is big enough), such that the cycle class map (1.2) is not surjective for such X. However, if one is interested only in quasi-projective counterexamples for the surjectivity of the map (1.2), one can produce more naive examples, for instance as a complement of some smooth hypersurfaces in a projective space. Hence we are interested to find an approximation of Chow groups and the étale cohomology of $BG_{\bar{k}}$ as a smooth and projective variety. In the case when the group G is finite, this is done in [3, Théorème 2.1]. In this section we give such an approximation for the groups we consider here, this construction is suggested by B. Totaro.

Proposition 4.1. *Let G be a compact Lie group as in (3.1). For all but finitely many primes p there exists a smooth and projective variety X_k over a finite field k with $char\, k = p$, an element $x_{4,k} \in H^4_{ét}(B(\mathbb{G}_m \times G_{\bar{k}}), \mathbb{Z}_\ell(2))$, invariant under the action of $Gal(\bar{k}/k)$ and a map $\tau : X_k \to B(\mathbb{G}_m \times G_k)$ in the category $\mathcal{H}.(k)$ such that*

- (i) *$y_{4,k} = \tau^* pr_2^* x_{4,k}$ is a non zero class in $H^4_{ét}(X_{\bar{k}}, \mathbb{Z}_\ell(2))/torsion$, where $pr_2 : \mathbb{G}_m \times G_k \to G_k$ is the projection on the second factor;*
- (ii) *the operation $Q_1(\bar{y}_{4,k})$ is non zero, where we write $\bar{y}_{4,k}$ for the image of $y_{4,k}$ in $H^4_{ét}(X_{\bar{k}}, \mathbb{Z}/\ell)$.*

Remark 4.2. For the purpose of this note, the proposition above is enough. See also [6] for a a general statement on the projective approximation of the cohomology of classifying spaces.

Theorem 1.1 now follows from the proposition above:

Proof of theorem 1.1.
For k a finite field and X_k as in the proposition above, we find a non-trivial class $y_{4,k}$ in its cohomology in degree 4 modulo torsion, which is not annihilated by the operation Q_1. This class can not be algebraic by proposition 2.2(iii). □

Proof of proposition 4.1.
We proceed in three steps. First, we construct a quasi-projective approximation in a family parametrized by $Spec\, \mathbb{Z}$. Then, for the geometric generic fibre we produce a projective approximation, by a topological argument. We finish the proof by specialization.

Step 1: construction of a family.
Let $\mathcal{G}$ be a split reductive group over $B = Spec\, \mathbb{Z}$ corresponding to G,

such a group exists by [SGA3] XXV 1.3. As B is an affine scheme of dimension 1, we can embed $\mathcal{G}$ as a closed subgroup of $GL_{d,B}$ for some d (see [SGA3] VI_B 13.2 and 13.5). Moreover, one can assume that $\mathcal{G} \hookrightarrow PGL_{d,B}$ such that this enbedding lifts to $\mathcal{H} = GL_{d,B}$ (e.g. using the embedding $GL_d \hookrightarrow PGL_{d+2}, A \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & A \end{pmatrix}$, up to changing d by $d+2$).

By a construction of [21, Remark 1.4] and [2, Lemme 9.2], there exists $n > 0$, a linear $\mathcal{H}$-representation $\mathcal{O}_B^{\oplus n}$ and an $\mathcal{H}$-invariant open subset $\mathcal{U} \subset \mathcal{O}_B^{\oplus n}$, which one can assume flat over B, such that the action of $\mathcal{H}$ is free on $\mathcal{U}$. Let $\mathcal{V}_N = \mathcal{O}_B^{\oplus Nn}$. Then the group $PGL_{n,B}$ acts on $\mathbb{P}(\mathcal{V}_N)$ and, taking N sufficiently large, one can assume that the action is free outside a subset S of high codimension $s \geq 4$.

By restriction, the group $\mathcal{G}$ acts on $\mathbb{P}(\mathcal{V}_N)$ as well, let $\mathcal{Y} = \mathbb{P}(\mathcal{V}_N)//\mathcal{G}$ be the GIT quotient for this action [15, 18]. The scheme $\mathcal{Y}$ is projective over B and we fix an embedding $\mathcal{Y} \subset \mathbb{P}_B^M$. Let

$$f : \mathcal{W} \to B \tag{4.2}$$

be the open set of $\mathcal{Y}$ corresponding to the quotient of the open set $\mathcal{U}$ as above where $\mathcal{G}$ acts freely. From the construction, $\mathcal{Y} - \mathcal{W}$ has high codimension in $\mathcal{Y}$.

For any point $b \in B$ with residue field $\kappa(b)$, the fibre $\mathcal{W}_b$ is a smooth quasi-projective variety and if N is big enough, we have isomorphisms by lifting $\mathcal{G}$ to $GL_{n,B}$ (cf. p. 263 in [21])

$$\mathcal{W}_b \cong (\mathbb{P}(\mathcal{V}_N) - S)_b/\mathcal{G}_b \cong ((\mathcal{V}_N - \{0\})/\mathbb{G}_m - S)_b)/\mathcal{G}_b \cong (\mathcal{V}_N - S')_b/(\mathbb{G}_m \times \mathcal{G})_b$$

where $S' = pr^{-1}S \cup \{0\}$ for the projection $pr : (\mathcal{V}_N - \{0\}) \to \mathbb{P}(\mathcal{V}_N)$. Hence we have isomorphisms

$$H^i(\mathcal{W}_b, \mathbb{Z}_\ell) \xrightarrow{\sim} H^i(B(\mathbb{G}_m \times \mathcal{G})_b, \mathbb{Z}_\ell) \text{ for } i \leq s, \ell \neq char\, \kappa(b), \tag{4.3}$$

induced by a natural map $\mathcal{W}_b \to B(\mathbb{G}_m \times \mathcal{G}_b)$ from the presentation (4.1).

Step 2: the generic fibre.
Let $Y = \mathcal{Y}_{\mathbb{C}}$ and $W = \mathcal{W}_{\mathbb{C}}$ be the geometric generic fibres of $\mathcal{Y}$ and $\mathcal{W}$ over B. Consider a general linear space L in $\mathbb{P}^M$ of codimension equal to $1 + dim(Y - W)$. Then $L \cap Y = L \cap W$ so $X := L \cap W$ is a smooth projective variety. Note that one can assume that L is defined over $\mathbb{Q}$.

By a version of the Lefschetz hyperplane theorem for quasi-projective varieties, established by Hamm (as a special case of Theorem II.1.2 in [4]), for $V \subset \mathbb{P}^M$ a closed complex subvariety of dimension d, not necessarily smooth, $Z \subset V$ a closed subset, and H a hyperplane in $\mathbb{P}^M$, if $V - (Z \cup H)$ is local complete intersection (e.g. $V - Z$ is smooth) then

$$\pi_i((V-Z) \cap H) \to \pi_i(V-Z)$$

is an isomorphism for $i < d-1$ and surjective for $i = d-1$. In particular, $H^i((V-Z) \cap H, \mathbb{Z}) \to H^i(V-Z, \mathbb{Z})$ is an isomorphism for $i < d-1$ and surjective for $i = d-1$ by the Whitehead theorem.

We then deduce that

$$H^i(X, R) \xrightarrow{\sim} H^i(B(\mathbb{G}_m \times G), R) \text{ for } i \leq s \text{ and } R = \mathbb{Z} \text{ or } \mathbb{Z}/n. \tag{4.4}$$

Hence $H^i_{\acute{e}t}(X, \mathbb{Z}/n) \xrightarrow{\sim} H^i_{\acute{e}t}(B(\mathbb{G}_m \times G), \mathbb{Z}/n), i \leq s$. Note that as the cohomology of BG is a direct factor in the cohomology of $B(\mathbb{G}_m \times G)$, we get that $x_4(G)$ (with the notations of the previous section) generates a direct factor isomorphic to $\mathbb{Z}_\ell$ in the cohomology group $H^4_{\acute{e}t}(X, \mathbb{Z}_\ell)$.

Step 3: specialization argument.
We can now specialize the construction above to obtain the statement over a finite field.

More precisely, one can find a dense open set $B' \subset B$ and a linear space $\mathcal{L} \subset \mathbb{P}^M_{B'}$ such that $\mathcal{L}_\mathbb{C} \simeq L$ and such that for any $b \in B'$ the fibre $\mathcal{X}_b$ of $\mathcal{X} = \mathcal{L} \cap \mathcal{Y}$ is smooth. Up to passing to an étale cover of B', one can assume that the inclusion $(\mathbb{Z}/\ell)^3 \subset G_\mathbb{C}$ from proposition 3.1 extends an inclusion $i : \mathcal{A} = (\mathbb{Z}/\ell)^3_{B'} \hookrightarrow \mathcal{G}_{B'}$ (cf. [SGA3] XI.5.8).

Let $b \in B'$ and let $k = \kappa(b)$. As the schemes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}/\mathcal{A}$ are smooth over B', we have the following commutative diagram, where the vertical maps are induced by the specialisation maps:

$$\begin{array}{ccccccc}
H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2)) & \longleftarrow & H^4_{\acute{e}t}(Y, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4_{\acute{e}t}(\mathcal{U}_\mathbb{C}/(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) & \xleftarrow{\simeq} & H^4_{\acute{e}t}(B(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) \\
\downarrow & & \downarrow & & \downarrow & & \| \\
H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2)) & \longleftarrow & H^4_{\acute{e}t}(\mathcal{Y}_{\bar{k}}, \mathbb{Z}_\ell(2)) & \longrightarrow & H^4_{\acute{e}t}(\mathcal{U}_{\bar{k}}/(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) & \xleftarrow{\simeq} & H^4_{\acute{e}t}(B(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell)
\end{array}$$

The left vertical map is an isomorphism since $\mathcal{X}$ is proper. Hence we get a class $y_{4,k} \in H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2))$, corresponding to $x_4(G) \in H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2))$. The map $H^4_{\acute{e}t}(Y, \mathbb{Z}_\ell(2)) \to H^4_{\acute{e}t}(X, \mathbb{Z}_\ell(2))$ is an isomorphism by step 2, so that $y_{4,k}$ comes from an element $x_{4,k} \in H^4_{\acute{e}t}(\mathcal{Y}_{\bar{k}}, \mathbb{Z}_\ell(2))$. Let $z_{4,k} \in H^4_{\acute{e}t}(B(\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell)$ be the image of $x_{4,k}$. From the diagram and proposition 3.1 we deduce that $Q_1(z_{4,k}) = Q_1Q_0(x_1x_2x_3) \neq 0$, hence $Q_1(\bar{y}_{4,k})$ is non zero as well. From the construction, the class $y_{4,k}$ generates a subgroup of $H^4_{\acute{e}t}(\mathcal{X}_{\bar{k}}, \mathbb{Z}_\ell(2))$, which is a direct factor isomorphic to $\mathbb{Z}_\ell$, and is Galois-invariant. Letting $X_k = \mathcal{X}_k$ this finishes the proof of the proposition.

□

Remark 4.3. We can also adapt the arguments of [3, Théorème 2.1] to produce projective examples with higher torsion non-algebraic classes, while in *loc.cit.* one constructs ℓ-torsion classes. Let $G(n)$ be the finite group $G(\mathbb{F}_{\ell^n})$, so that we have

$$\varprojlim H^*_{\acute{e}t}(BG(n), \mathbb{Z}_\ell) = H^*_{\acute{e}t}(BG_{\bar{k}}, \mathbb{Z}_\ell).$$

Then, following the construction in *loc.cit.* one gets

> *For any $n > 0$, there exits a positive integer i_n and a Godeaux-Serre variety $X_{n,\bar{k}}$ for the finite group $G(n)$ such that*
> (1) $x \in H^4_{\acute{e}t}(X_{n,\bar{k}}; \mathbb{Z}_\ell(2))$ *generates* $\mathbb{Z}/\ell^{n'}$ *for some* $n' \geq n$;
> (2) x *is not in the image of the cycle class map (1.1).*

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