UNIFORM BOUNDS FOR OBSTRUCTIONS TO THE INTEGRAL TATE CONJECTURE

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Abstract. Assuming natural variational realization conjectures, we give uniform bounds for the obstruction to the integral Tate conjecture in 1-dimensional families of algebraic varieties over an infinite finitely generated field.

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1. Introduction

For an abelian group A, write $A_{\mathrm{tors}} \subset A$ for its torsion subgroup and $A \twoheadrightarrow A^{\mathrm{free}} := A/A_{\mathrm{tors}}$ for its maximal torsion-free quotient. For an algebraic group G, let $G^\circ \subset G$ denote its neutral component and $G \twoheadrightarrow \pi_0(G) := G/G^\circ$ its group of connected components.

A variety over a field k is a separated scheme of finite type over k.

In this paper k will denote an infinite field of characteristic $p \geq 0$, finitely generated over its prime subfield. We fix a separable closure $k \hookrightarrow \bar{k}$ and write $\pi_1(k) =$Gal$(\bar{k}|k)$ for the absolute Galois group.

1.1. **Tate conjectures.** Let X be a smooth projective variety over k. For every integer $i \geq 0$, let $\mathrm{CH}^i(X)$ denote the group of algebraic cycles of codimension i on X modulo rational equivalence, and for every ring R, set $\mathrm{CH}^i(X)_R := \mathrm{CH}^i(X) \otimes_{\mathbb{Z}} R$. For a prime $\ell \neq p$, set

$$V_{\mathbb{Z}_\ell} := \mathrm{H}^{2i}(X_{\bar{k}}, \mathbb{Z}_\ell(i)).$$

Let $G_\ell \subset \mathrm{GL}(V_{\mathbb{Q}_\ell})$ denote the Zariski-closure of the image of $\pi_1(k)$ acting on $V_{\mathbb{Q}_\ell} := V_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and let

$$\widetilde{V}_{\mathbb{Q}_\ell} := (V_{\mathbb{Q}_\ell})^{G_\ell^\circ} \subset V_{\mathbb{Q}_\ell}$$

denote the $\mathbb{Q}_\ell$-vector space of Tate classes. The cycle class map $c_\ell : \mathrm{CH}^i(X_{\bar{k}}) \to V_{\mathbb{Z}_\ell}$ for $\mathbb{Z}_\ell$-étale cohomology fits into the following canonical Cartesian diagram

(1)
$$\begin{tikzcd}
\mathrm{CH}^i(X_{\bar{k}}) \arrow[r] \arrow[dd] \arrow[rrrr, bend left, "c_\ell"] & \mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Z}_\ell} \arrow[r, two heads] \arrow[dd] & V^a_{\mathbb{Z}_\ell} \arrow[r, hook] \arrow[d] & \widetilde{V}_{\mathbb{Z}_\ell} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & V_{\mathbb{Z}_\ell} \arrow[d] \\
 & & V^{\mathrm{free},a}_{\mathbb{Z}_\ell} \arrow[r, hook] \arrow[d, hook] & \widetilde{V}^{\mathrm{free}}_{\mathbb{Z}_\ell} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & V^{\mathrm{free}}_{\mathbb{Z}_\ell} \arrow[d, hook] \\
\mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Q}} \arrow[r] & \mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Q}_\ell} \arrow[r, two heads] & V^a_{\mathbb{Q}_\ell} \arrow[r, hook] & \widetilde{V}_{\mathbb{Q}_\ell} \arrow[r, hook] & V_{\mathbb{Q}_\ell},
\end{tikzcd}$$

where $V^a_{\mathbb{Z}_\ell}$ (resp. $V^a_{\mathbb{Q}_\ell}$) is the image of the cycle class map $c_\ell \otimes \mathbb{Z}_\ell : \mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Z}_\ell} \to V_{\mathbb{Z}_\ell}$ (resp. $c_\ell \otimes \mathbb{Q}_\ell$) and where $\widetilde{V}_{\mathbb{Z}_\ell}$ and $\widetilde{V}^{\mathrm{free}}_{\mathbb{Z}_\ell}$ are defined by the rightmost Cartesian squares of the diagram.

The (classical) rational $\mathbb{Q}_\ell$-Tate conjecturefor codimension i cycles on X [Ta65]

$$\mathrm{Tate}_{\mathbb{Q}_\ell}(X,i) \;\; V^a_{\mathbb{Q}_\ell} = \widetilde{V}_{\mathbb{Q}_\ell}$$

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admits the following integral variants:

$$\begin{array}{lll} \mathrm{Tate}^{\mathrm{free}}_{\mathbb{Z}_\ell}(X,i) & V^{\mathrm{free},a}_{\mathbb{Z}_\ell}=\widetilde{V}^{\mathrm{free}}_{\mathbb{Z}_\ell} & \text{(Integral Tate conjecture modulo torsion);}\\ \mathrm{Tate}_{\mathbb{Z}_\ell}(X,i) & V^{a}_{\mathbb{Z}_\ell}=\widetilde{V}_{\mathbb{Z}_\ell} & \text{(Integral Tate conjecture).}\end{array}$$

While, tautologically,

$$\mathrm{Tate}_{\mathbb{Z}_\ell}(X,i)\Rightarrow \mathrm{Tate}^{\mathrm{free}}_{\mathbb{Z}_\ell}(X,i)\Rightarrow \mathrm{Tate}_{\mathbb{Q}_\ell}(X,i),$$

it is known that, in general, the converse implications fail (see e.g. [CTS10, AtH62] for an example of the failure of $\mathrm{Tate}_{\mathbb{Z}_\ell}(X,i)$ and [CTS10, Ko90, To13] for examples of the failure of $\mathrm{Tate}^{\mathrm{free}}_{\mathbb{Z}_\ell}(X,i)$).

The aim of this note is to analyze the obstructions to $\mathrm{Tate}_{\mathbb{Z}_\ell}(X,i)$, $\mathrm{Tate}^{\mathrm{free}}_{\mathbb{Z}_\ell}(X,i)$ when X varies in family. Our arguments provide a new application of the structure theorem of the degeneration locus of ℓ-adic local systems of [CT13] (see Fact A), in the spirit of [CC20, C23].

Before considering the variational setting, we make some elementary remarks. By definition, the obstructions to $\mathrm{Tate}_{\mathbb{Q}_\ell}(X,i)$, $\mathrm{Tate}^{\mathrm{free}}_{\mathbb{Z}_\ell}(X,i)$, $\mathrm{Tate}_{\mathbb{Z}_\ell}(X,i)$ are, respectively:

$$\widetilde{C}_{\mathbb{Q}_\ell}:=\widetilde{V}_{\mathbb{Q}_\ell}/V^a_{\mathbb{Q}_\ell},\ \ \widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell}:=\widetilde{V}^{\mathrm{free}}_{\mathbb{Z}_\ell}/V^{\mathrm{free},a}_{\mathbb{Z}_\ell},\ \ \widetilde{C}_{\mathbb{Z}_\ell}:=\widetilde{V}_{\mathbb{Z}_\ell}/V^a_{\mathbb{Z}_\ell}.$$

1.1.1. $\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell}$ *versus* $\widetilde{C}_{\mathbb{Z}_\ell}$. The short exact sequence

$$0\to (V_{\mathbb{Z}_\ell})_{\mathrm{tors}}/(V^a_{\mathbb{Z}_\ell})_{\mathrm{tors}}\to \widetilde{C}_{\mathbb{Z}_\ell}\to \widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell}\to 0 \tag{2}$$

realizes $\widetilde{C}_{\mathbb{Z}_\ell}$ an extension of $\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell}$ by a finite group which is a quotient of $(V_{\mathbb{Z}_\ell})_{\mathrm{tors}}$. As $(V_{\mathbb{Z}_\ell})_{\mathrm{tors}}$ is constant in family, the problems of bounding uniformly $\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell}$ and $\widetilde{C}_{\mathbb{Z}_\ell}$ are essentially equivalent.

1.1.2. $\widetilde{C}_{\mathbb{Q}_\ell}$ *versus* $\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell}$. From $\widetilde{C}_{\mathbb{Q}_\ell}=\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ and the short exact sequence (2), one has the folllowing equivalences

$$\mathrm{Tate}_{\mathbb{Q}_\ell}(X,i)\Leftrightarrow (\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell})_{\mathrm{tors}}=\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell}\Leftrightarrow (\widetilde{C}_{\mathbb{Z}_\ell})_{\mathrm{tors}}=\widetilde{C}_{\mathbb{Z}_\ell}$$

and, in case they hold, (2) reads

$$0\to (V_{\mathbb{Z}_\ell})_{\mathrm{tors}}/(V^a_{\mathbb{Z}_\ell})_{\mathrm{tors}}\to (\widetilde{C}_{\mathbb{Z}_\ell})_{\mathrm{tors}}\to (\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell})_{\mathrm{tors}}\to 0. \tag{3}$$

So that, assuming $\mathrm{Tate}_{\mathbb{Q}_\ell}(X,i)$, the obstructions we are interested in are $(\widetilde{C}_{\mathbb{Z}_\ell})_{\mathrm{tors}}$, $(\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell})_{\mathrm{tors}}$. The obstruction $(\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell})_{\mathrm{tors}}$ can be described without involving the $\mathbb{Z}_\ell$-module $\widetilde{V}^{\mathrm{free}}_{\mathbb{Z}_\ell}$ of Tate classes. Indeed, writing

$$C^{\mathrm{free}}_{\mathbb{Z}_\ell}:=V^{\mathrm{free}}_{\mathbb{Z}_\ell}/V^{\mathrm{free},a}_{\mathbb{Z}_\ell},$$

it follows from the short exact sequence

$$0\to C^{\mathrm{free}}_{\mathbb{Z}_\ell}\to \widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell}\to V^{\mathrm{free}}_{\mathbb{Z}_\ell}/\widetilde{V}^{\mathrm{free}}_{\mathbb{Z}_\ell}\to 0$$

and the fact that $V^{\mathrm{free}}_{\mathbb{Z}_\ell}/\widetilde{V}^{\mathrm{free}}_{\mathbb{Z}_\ell}$ is torsion-free that

$$(C^{\mathrm{free}}_{\mathbb{Z}_\ell})_{\mathrm{tors}}=(\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}_\ell})_{\mathrm{tors}}.$$

1.2. Let now S be a smooth, geometrically connected variety over k, with generic point η, and $f:X\to S$ a smooth projective morphism. For $s\in S$, denote by a subscript $(-)_s$ the various modules attached to X_s introduced above (*e.g.* $V_{\mathbb{Z}_\ell,s}:=\mathrm{H}^{2i}(X_{\bar{s}},\mathbb{Z}_\ell(i))$, $V^a_{\mathbb{Z}_\ell,s}:=\mathrm{im}[\mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Z}_\ell}\to V_{\mathbb{Z}_\ell,s}]$ *etc.*). One would like to investigate how

$$\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell,s}:=|(\widetilde{C}_{\mathbb{Z}_\ell,s})_{\mathrm{tors}}|$$

vary with $s\in|S|$. In particular, the vanishing of the obstruction group $(\widetilde{C}_{\mathbb{Z}_\ell,s})_{\mathrm{tors}}$ reads as $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell,s}=1$.

1.2.1. Assume first $p = 0$. The following statement is predicted by the main conjecture of [C23]. For every integer $d \geq 1$, let $|S|^{\leq d} \subset |S|$ denote the set of all closed points $s \in |S|$ with residue degree $[k(s) : k] \leq d$.

Conjecture 1. *For every integer $d \geq 1$, one has*

$$\widetilde{\mathrm{Ob}}^{\leq d}_{\mathbb{Z}_\ell} := \sup\{\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell,s} \mid s \in |S|^{\leq d}\} < +\infty$$

and $\widetilde{\mathrm{Ob}}^{\leq d}_{\mathbb{Z}_\ell} = 1$, $\ell \gg 0$.

Our first main result is that Conjecture 1 holds when S is a curve *modulo* some reasonable variational realization conjecture, which we discuss now.

- **Singular cohomology**: Fix an embedding $\infty : k \hookrightarrow \mathbb{C}$, let $(-)_\infty$ denote the base-change functor along $\mathrm{Spec}(\mathbb{C}) \overset{\infty}{\to} \mathrm{Spec}(k)$ and $(-)^{\mathrm{an}}$ the analytification functor from varieties over $\mathbb{C}$ to complex analytic spaces. For every $s_\infty \in S_\infty(\mathbb{C})$ the cycle class maps for singular cohomology

$$c : \mathrm{CH}^i(X_\infty)_{\mathbb{Q}} \to \mathrm{H}^{2i}(X^{\mathrm{an}}_\infty, \mathbb{Q}(i)), \quad c_{s_\infty} : \mathrm{CH}^i(X_{s_\infty})_{\mathbb{Q}} \to \mathrm{H}^{2i}(X^{\mathrm{an}}_{s_\infty}, \mathbb{Q}(i))$$

fit into a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^i(X_\infty)_{\mathbb{Q}} \arrow[rr, "|_{X_{\infty,s}}"] \arrow[d, "c"'] & & \mathrm{CH}^i(X_{s_\infty})_{\mathbb{Q}} \arrow[d, "c_{s_\infty}"] \\
\mathrm{H}^{2i}(X^{\mathrm{an}}_\infty, \mathbb{Q}(i)) \arrow[r, "\epsilon"] & \mathrm{H}^0(S^{\mathrm{an}}_\infty, R^{2i} f^{\mathrm{an}}_{\infty *}\mathbb{Q}(i)) \arrow[r, hook] & \mathrm{H}^{2i}(X^{\mathrm{an}}_{s_\infty}, \mathbb{Q}(i)),
\end{tikzcd}$$

where $\epsilon : \mathrm{H}^{2d}(X^{\mathrm{an}}_\infty, \mathbb{Q}(i)) \twoheadrightarrow E^{0,i}_\infty \hookrightarrow E^{0,i}_2 = \mathrm{H}^0(S^{\mathrm{an}}_\infty, R^{2i} f^{\mathrm{an}}_{\infty *}\mathbb{Q}(i))$ is the edge morphism from the Leray spectral sequence for $f^{\mathrm{an}}_\infty : X^{\mathrm{an}}_\infty \to S^{\mathrm{an}}_\infty$.

$\mathrm{VSing}^0(f_\infty, i)$ For every $s_\infty \in S_\infty(\mathbb{C})$ and $\alpha_{s_\infty} \in \mathrm{H}^0(S^{\mathrm{an}}_\infty, R^{2i} f^{\mathrm{an}}_{\infty *}\mathbb{Q}(i)) \subset \mathrm{H}^{2i}(X^{\mathrm{an}}_{s_\infty}, \mathbb{Q}(i))$ the following properties are equivalent:

1) $\alpha_{s_\infty} \in \mathrm{im}[c_{s,\mathbb{Q}} : \mathrm{CH}^i(X_{s_\infty})_{\mathbb{Q}} \to \mathrm{H}^{2i}(X^{\mathrm{an}}_{s_\infty}, \mathbb{Q}(i))]$;

2) there exists $\widetilde{\alpha} \in \mathrm{CH}^i(X_\infty)_{\mathbb{Q}}$ such that $c_{s_\infty}(\widetilde{\alpha}|_{X_{s_\infty}}) = \alpha_{s_\infty}$.

Though it does not involve Hodge classes, the statement $\mathrm{VSing}^0(f_\infty, i)$ is often referred to as the variational Hodge conjecture for codimension i cycles because, by the fixed part theorem, it follows from the Hodge conjecture for any smooth compactification of X_∞ - see *e.g.* [CS13, §3.1] for details and an equivalent formulation using de Rham cohomology. *A priori* the statement $\mathrm{VSing}^0(f_\infty, i)$ is not preserved by base-change along finite covers of smooth varieties while the obstructions $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell,s}$, $s \in S$ are. So we will rather consider the following "stabilized" variant $\mathrm{VSing}(f_\infty, i)$. For finite covers $S''_\infty \to S'_\infty \to S_\infty$ of smooth varieties, consider the notation in the base-change diagram:

$$\begin{tikzcd}
X''_\infty \arrow[r] \arrow[d, "f''_\infty"'] \arrow[dr, phantom, "\square"] & X'_\infty \arrow[r] \arrow[d, "f'_\infty"'] \arrow[dr, phantom, "\square"] & X_\infty \arrow[d, "f_\infty"] \\
S''_\infty \arrow[r] & S'_\infty \arrow[r] & S_\infty.
\end{tikzcd}$$

$\mathrm{VSing}(f_\infty, i)$ There exists a finite cover $S'_\infty \to S_\infty$ of smooth varieties over $\mathbb{C}$ such that for every finite cover $S''_\infty \to S'_\infty$ of smooth varieties over $\mathbb{C}$, $\mathrm{VSing}^0(f''_\infty, i)$ holds.

- **Étale $\mathbb{Q}_\ell$-cohomology**: The following is the $\mathbb{Q}_\ell$-étale counterpart of $\mathrm{VSing}^0(f_\infty, i)$:

$\mathrm{VEt}^0_{\mathbb{Q}_\ell}(f, i)$ For every $s \in |S|$ and $\alpha_s \in \mathrm{H}^0(S_{\bar{k}}, R^{2i} f_* \mathbb{Q}_\ell(i)) \subset \mathrm{H}^{2i}(X_{\bar{s}}, \mathbb{Q}_\ell(i))$ the following properties are equivalent:

1) $\alpha_s \in \mathrm{im}[c_{X_{\bar{s}},\ell} : \mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Q}} \to \mathrm{H}^{2i}(X_{\bar{s}}, \mathbb{Q}_\ell(i))]$;

2) there exists $\widetilde{\alpha} \in \mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Q}}$ such that $c_{X_{\bar{s}},\ell}(\widetilde{\alpha}|_{X_{\bar{s}}}) = \alpha_s$.

One could also consider the seemingly weaker variant $\mathrm{WVEt}^0_{\mathbb{Q}_\ell}(f, i)$ where $\mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Q}}$, $\mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Q}}$ are replaced with $\mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Q}_\ell}$, $\mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Q}_\ell}$, and the stabilized variants $\mathrm{WVEt}_{\mathbb{Q}_\ell}(f, i)$, $\mathrm{VEt}_{\mathbb{Q}_\ell}(f, i)$. Note that the statements $\mathrm{WVEt}^0_{\mathbb{Q}_\ell}(f, i)$, $\mathrm{VEt}^0_{\mathbb{Q}_\ell}(f, i)$ also make sense when $p > 0$.

Proposition 2. *If $p=0$, one has*
$$\mathrm{WVEt}^0_{\mathbb{Q}_\ell}(f,i) \Leftrightarrow \mathrm{VEt}^0_{\mathbb{Q}_\ell}(f,i) \Leftrightarrow \mathrm{VSing}^0(f_\infty,i).$$
In general, one always has $\mathrm{VEt}^0_{\mathbb{Q}_\ell}(f,i) \Rightarrow \mathrm{WVEt}^0_{\mathbb{Q}_\ell}(f,i)$ and $\mathrm{Tate}_{\mathbb{Q}_\ell}(X_\eta,i) \Rightarrow \mathrm{WVEt}_{\mathbb{Q}_\ell}(f,i)$.

We will give a proof of this proposition in section 3.1.3.
In particular, when $p=0$, $\mathrm{VSing}^0(f_\infty,i)$ is independent of the embedding $\infty : k \hookrightarrow \mathbb{C}$ and $\mathrm{WVEt}^0_{\mathbb{Q}_\ell}(f,i)$, $\mathrm{VEt}^0_{\mathbb{Q}_\ell}(f,i)$ are independent of the prime ℓ.

We can now state our first main result.

Theorem A. *Assume S is a curve and VSing(f_∞,i) holds for one (equivalently every) embedding $\infty : k \hookrightarrow \mathbb{C}$. Then, for every integer $d\geq 1$, one has $\widetilde{\mathrm{Ob}}^{\leq d}_{\mathbb{Z}_\ell} < +\infty$ and $\widetilde{\mathrm{Ob}}^{\leq d}_{\mathbb{Z}_\ell} = 1$ for $\ell \gg 0$ (depending on d).*

1.2.2. Assume now $p>0$. One has a variant of Theorem A for $d=1$ but it is slightly more technical. To state it, one has to make a mild assumption on the $\mathbb{Q}_\ell$-local system $\mathcal{V}_{\mathbb{Q}_\ell} := R^{2i}f_*\mathbb{Q}_\ell(i)$, namely that it is GLU - see Subsection 2.2.1.2 for the definition. One also needs a substitute for $\mathrm{VSing}(f_\infty,i)$. According to Proposition 2, a first substitute is $\mathrm{WVEt}_{\mathbb{Q}_\ell}(f,i)$. Another natural substitute is the variational realization conjecture in crystalline cohomology $\mathrm{VCrys}(f,i)$. This is more subtle. Indeed, as crystalline cohomology is only well-behaved over a perfect residue field, one has first to spread out all the involved data over a finite base field. Another difficulty is that the proof of Theorem A heavily relies on Artin's comparison isomorphism bewteen étale and singular cohomology. But there is no such a direct functorial comparison isomorphism between crystalline and étale cohomology; to remedy this, one has to invoke a weak form - $\mathrm{CrysEt}_{\mathbb{Q}_\ell}(f,i)$ of the motivic conjecture predicting that homological and numerical equivalence should coincide (combined with a theorem of Ambrosi - see Fact 12).

We now state $\mathrm{VCrys}(f,i)$ and $\mathrm{CrysEt}_{\mathbb{Q}_\ell}(f,i)$. Let F denote the algebraic closure of $\mathbb{F}_p$ in k and let $\mathscr{K}$ be a smooth, separated, geometrically connected scheme over F with generic point $\eta_{\mathscr{K}} : \mathrm{Spec}(k) \to \mathscr{K}$, let $\mathcal{S} \to \mathscr{K}$ be a smooth, separated and geometrically connected morphism and $f : \mathcal{X} \to \mathcal{S}$ a smooth proper morphism fitting in the following Cartesian diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "f"] & \mathcal{S} \arrow[r] & \mathscr{K} \\
X \arrow[u] \arrow[r, "f"'] \arrow[ur, phantom, "\square"] & S \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & k \arrow[u, "\eta_{\mathscr{K}}"']
\end{tikzcd}$$

Let K denote the fraction field of the ring W of Witt vectors of F. For a F-scheme $\mathcal{Z}$, write $\mathrm{H}^i_{\mathrm{crys}}(\mathcal{Z}) := \mathrm{H}^i_{\mathrm{crys}}(\mathcal{Z}/W)_K$ for the crystalline cohomology with K-coefficients and
$$c_{\mathrm{crys}} : \mathrm{CH}^i(\mathcal{Z})_{\mathbb{Q}} \to \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{Z})$$
for the cycle class map. For every $t\in|\mathcal{S}|$ the cycle class maps
$$c_{\mathrm{crys}} : \mathrm{CH}^i(\mathcal{X}) \to \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}),\ \ c_{\mathrm{crys},t} : \mathrm{CH}^i(\mathcal{X}_t) \to \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}_t)$$
fit into a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^i(\mathcal{X})_{\mathbb{Q}} \arrow[rr, "|_{\mathcal{X}_t}"] \arrow[d, "c_{\mathrm{crys}}"'] & & \mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} \arrow[d, "c_{\mathrm{crys},t}"] \\
\mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}) \arrow[r, "\epsilon"] & \mathrm{H}^0(\mathcal{S}, R^{2i}f_{\mathrm{crys},*}\mathcal{O}_{\mathcal{X}/W})_K \arrow[r, hook] & \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}_t),
\end{tikzcd}$$

where $\epsilon : \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}) \twoheadrightarrow E^{0,i}_\infty \hookrightarrow \mathrm{H}^0(\mathcal{S}, R^{2i}f_{\mathrm{crys},*}\mathcal{O}_{\mathcal{X}/W})_K$ is, again, the edge morphism from the Leray spectral sequence for $f:\mathcal{X}\to\mathcal{S}$ in crystalline cohomology - see [M23, §1] and the references therein for details. The following is the crystalline analogue of $\mathrm{VSing}^0(f_\infty,i)$, $\mathrm{VEt}^0_{\mathbb{Q}_\ell}(f,i)$ [M23, Conj. 0.1].

$\mathrm{VCrys}^0(f,i)$ For every $t\in|\mathcal{S}|$ and $\alpha_t \in \mathrm{H}^0(\mathcal{S}, R^{2i}f_{\mathrm{crys},*}\mathcal{O}_{\mathcal{X}/W})_{\mathbb{Q}} \subset \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}_t)$ the following properties are equivalent:

1) $\alpha_t \in \mathrm{im}[c_{\mathrm{crys},t} : \mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} \to \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}_t)]$;

2) there exists $\widetilde{\alpha} \in \mathrm{CH}^i(\mathcal{X})_{\mathbb{Q}}$ such that $c_{\mathrm{crys},t}(\widetilde{\alpha}|_{\mathcal{X}_t}) = \alpha_t$.

As before, let $\mathrm{VCrys}(f,i)$ denote its stabilized variant.

Also, consider the following statement

$\mathrm{CrysEt}_{\mathbb{Q}_\ell}(f,i)$ For every $t \in |\mathcal{S}|$, the kernel of the cycle class maps

$$c_{\mathrm{crys},t} : \mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} \to \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}_t), \quad c_{\ell,t} : \mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} \to \mathrm{H}^{2i}(\mathcal{X}_{\bar{t}}, \mathbb{Q}_\ell)$$

coincide,

which follows from the standard conjecture predicting that homological and numerical equivalences should coincide, which, in turn, is a consequence of the conjecture predicting that the category of effective motives should be abelian semisimple [J92].

We can now state the analogue of Theorem A when $p > 0$.

Theorem B. *Assume S is a curve, $\mathcal{V}_{\mathbb{Q}_\ell}$ is GLU and either (i)* $\mathrm{WVEt}_{\mathbb{Q}_\ell}(f,i)$ *or (ii)* $\mathrm{VCrys}(f,i)$+$\mathrm{CrysEt}_{\mathbb{Q}_\ell}(f,i)$ *holds. Then, one has $\widetilde{\mathrm{Ob}}^{\leq 1}_{\mathbb{Z}_\ell} < +\infty$.*

Remark 3. We do not know if, under the assumptions of Theorem B, $\widetilde{\mathrm{Ob}}^{\leq 1}_{\mathbb{Z}_\ell} = 0$, $\ell \gg 0$.

1.2.3. *Unramified cohomology.* When $i = 2$, $(\widetilde{C}_{\mathbb{Z}_\ell,s})_{\mathrm{tors}}$ can be described in terms of degree 3 unramified cohomology. More precisely, set $C_{\mathbb{Z}_\ell,s} := V_{\mathbb{Z}_\ell}/V^a_{\mathbb{Z}_\ell,s}$. From the short exact sequence

$$0 \to \widetilde{C}_{\mathbb{Z}_\ell,s} \to C_{\mathbb{Z}_\ell,s} \to V_{\mathbb{Z}_\ell,s}/\widetilde{V}_{\mathbb{Z}_\ell,s} \to 0$$

and the fact that $V_{\mathbb{Z}_\ell,s}/\widetilde{V}_{\mathbb{Z}_\ell,s}$ is torsion-free, one has $(\widetilde{C}_{\mathbb{Z}_\ell,s})_{\mathrm{tors}} = (C_{\mathbb{Z}_\ell,s})_{\mathrm{tors}}$. If $i = 2$, [CTK13, Thm. 2.2] states that $(C_{\mathbb{Z}_\ell,s})_{\mathrm{tors}}$ is isomorphic to

$$\mathrm{H}^3_{\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))_{\mathrm{ndiv}} \overset{def}{=} \mathrm{coker}[\mathrm{H}^3_{\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))_{\mathrm{div}} \to \mathrm{H}^3_{\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))].$$

Here for an abelian group A, we let $A_{\mathrm{div}} \subset A$ denote its maximal divisible subgroup.

Hence Theorem A and Theorem B for $i = 2$ imply:

Corollary 4. *Assume S is a curve.*

(1) Assume $p = 0$ and $\mathrm{VSing}(f_\infty, i)$ *for some embedding $\infty : k \hookrightarrow \mathbb{C}$ holds. Then, for every integer $d \geq 1$,*

$$\sup\{|\mathrm{H}^3_{\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))_{\mathrm{ndiv}}| \mid s \in |S|^{\leq d}\}| < +\infty,$$

and $\mathrm{H}^3_{\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))_{\mathrm{ndiv}} = 0$*, $s \in |S|^{\leq d}$ for $\ell \gg 0$ (depending on d).*

(2) Assume $p > 0$, $\mathcal{V}_{\mathbb{Q}_\ell}$ is GLU and either (i) $\mathrm{WVEt}_{\mathbb{Q}_\ell}(f,i)$ *or (ii)* $\mathrm{VCrys}(f,i)+\mathrm{CrysEt}_{\mathbb{Q}_\ell}(f,i)$ *holds. Then,*

$$\sup\{|\mathrm{H}^3_{\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))_{\mathrm{ndiv}}| \mid s \in S(k)\}| < +\infty,$$

and $\mathrm{H}^3_{nr}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))_{\mathrm{ndiv}} = 0$*, $s \in S(k)$ for $\ell \gg 0$.*

For integers $a \geq 0$, b, c and $A_\ell = \mathbb{Z}_\ell, \mathbb{Q}_\ell, \mathbb{Q}_\ell/\mathbb{Z}_\ell$ *etc.*, Schreieder introduces refined unramified cohomology groups $\mathrm{H}^a_{c,\mathrm{nr}}(X_{\bar{s}}, A_\ell(b))$ [S23, §1.2] which, when $c = 0$, coincide with the usual unramified cohomology groups. By [S23, Thm. 1.8], for every integer $i \geq 0$ one has:

$$(\widetilde{C}_{\mathbb{Z}_\ell,s})_{\mathrm{tors}} \simeq \mathrm{H}^{2i-1}_{i-2,\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))_{\mathrm{ndiv}} \overset{def}{=} \mathrm{coker}[\mathrm{H}^{2i-1}_{i-2,\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))_{\mathrm{div}} \to \mathrm{H}^{2i-1}_{i-2,\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))].$$

So, Corollary 4 holds more generally with $\mathrm{H}^3_{\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))_{\mathrm{ndiv}}$ replaced by $\mathrm{H}^{2i-1}_{i-2,\mathrm{nr}}(X_{\bar{s}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i))_{\mathrm{ndiv}}$.

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In Section 2.1 we review basic properties of cycle class maps for étale $\mathbb{Z}_\ell$-cohomology in families, introduce the notion of $\mathcal{V}_{\mathbb{Q}_\ell}$-generic points and describe the general strategy for the proof of Theorem A and Theorem B. In Section 3, we inject comparison with singular cohomology - Subsection 3.1, to prove Proposition 2 and conclude the proofs of Theorem A, and with crystalline cohomology - Subsection 3.2, to conclude the proof of Theorem B. In Subsection 3.1.5, we also explain how to derive from Theorem A its variant in the setting of the integral Hodge conjecture.

2. Étale cycle class maps in families and global strategy

2.1. **Étale $\mathbb{Z}_\ell$-local systems.** Let S be a smooth, geometrically connected variety over k. For every $s\in S$, fix a geometric point $\bar{s}$ over it and an étale path $\alpha_{\bar{s}}:(-)_{\bar{s}}\tilde{\to}(-)_{\bar{\eta}}$. In particular, for every $\mathbb{Z}_\ell$-local system $\mathcal{V}_{\mathbb{Z}_\ell}$ on S, one identifies $\mathcal{V}_{\mathbb{Z}_\ell,\bar{s}}\tilde{\to}\mathcal{V}_{\mathbb{Z}_\ell,\bar{\eta}}$ equivariantly with respect to the isomorphism of étale fundamental groups $\pi_1(S,\bar{s})\tilde{\to}\pi_1(S,\bar{\eta})$, $\gamma\mapsto\alpha_{\bar{s}}\gamma\alpha_{\bar{s}}^{-1}$. As a result, we will in general omit fiber functors from our notation and simply write

$$V_{\mathbb{Z}_\ell}:=\mathcal{V}_{\mathbb{Z}_\ell,\bar{s}}\tilde{\to}\mathcal{V}_{\mathbb{Z}_\ell,\bar{\eta}},\quad V_{\mathbb{Q}_\ell}:=V_{\mathbb{Z}_\ell}\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell.$$

Let $f:X\to S$ be a smooth projective morphism.

2.1.1. *Notational conventions.* Consider the $\mathbb{Z}_\ell$-étale local system $\mathcal{V}_{\mathbb{Z}_\ell}:=R^{2i}f_*\mathbb{Z}_\ell(i)$ on S. Let $G_\ell\subset\mathrm{GL}(V_{\mathbb{Q}_\ell})$ denote the Zariski-closure of the image of $\pi_1(S)$ acting on $V_{\mathbb{Q}_\ell}$; let also $\overline{G}_\ell\subset G_\ell$ and, for every $s\in S$, $G_{\ell,s}\subset G_\ell$ denote the Zariski closure of the images of $\pi_1(S_{\bar{k}})$ and $\pi_1(s)$ acting on $V_{\mathbb{Q}_\ell}$ by restriction along the functorial morphisms $\pi_1(S_{\bar{k}})\to\pi_1(S)$ and $\pi_1(s)\to\pi_1(S)$ respectively (in particular $G_{\ell,\eta}=G_\ell$). As S is geometrically connected over k, the functorial sequence

$$1\to\pi_1(S_{\bar{k}})\to\pi_1(S)\to\pi_1(k)\to 1$$

is exact, hence $\overline{G}_\ell\subset G_\ell$ is a normal subgroup, and for every closed point $s\in|S|$, one has $G_\ell^\circ=\overline{G}_\ell^\circ G_{\ell,s}^\circ$.

2.1.2. *Specialization and extension of algebraically closed fields.* We recall the following two properties of the cycle class map for étale $\mathbb{Z}_\ell$-cohomology.

2.1.2.1. *Compatibility with specialization of algebraic cycles.* For every $s\in S$, one has a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^i(X_{\bar{k}}) \arrow[r, "|_{X_{\bar{\eta}}}"] \arrow[d, "|_{X_{\bar{s}}}"'] & \mathrm{CH}^i(X_{\bar{\eta}}) \arrow[d, "c_{\ell,\eta}"] \arrow[dl, "sp_{\eta,s}"] \\
\mathrm{CH}^i(X_{\bar{s}}) \arrow[r, "c_{\ell,s}"'] & V_{\mathbb{Z}_\ell}
\end{tikzcd}$$

(see [F98, § 20.3, Ex. 20.3.1 and 20.3.5]).

2.1.2.2. *"Invariance" under extension of algebraically closed field.* Let $\Omega\hookrightarrow\Omega'$ be an extension of algebraically closed fields of characteristic $\neq\ell$ and let Y be a smooth proper variety over Ω. Consider the canonical commutative square

$$\begin{tikzcd}
\mathrm{CH}^i(Y) \arrow[r, "c_\ell"] \arrow[d, "|_{Y_{\Omega'}}"'] & \mathrm{H}^{2i}(Y,\mathbb{Z}_\ell(i)) \arrow[d, "\simeq"] \\
\mathrm{CH}^i(Y_{\Omega'}) \arrow[r, "c_\ell"] & \mathrm{H}^{2i}(Y_{\Omega'},\mathbb{Z}_\ell(i)).
\end{tikzcd}$$

Then[1],

$$\mathrm{im}[c_\ell\circ -|_{Y_{\Omega'}}]:\mathrm{CH}^i(Y)\to\mathrm{H}^{2i}(Y_{\Omega'},\mathbb{Z}_\ell(i))=\mathrm{im}[c_\ell:\mathrm{CH}^i(Y_{\Omega'})\to\mathrm{H}^{2i}(Y_{\Omega'},\mathbb{Z}_\ell(i))].$$

In particular, $V^a_{\mathbb{Z}_\ell,s}$, $V^{\mathrm{free},a}_{\mathbb{Z}_\ell,s}$ *etc.* are independent of the geometric point $\bar{s}$ over s.

[1]In fact, a cycle $\xi\in\mathrm{CH}^i(Y_{\Omega'})$ is defined over a finitely generated algebraically closed field $\Omega''\subset\Omega'$. One could then find a smooth and proper model of Y over a small affine scheme U over Ω with generic point Ω'' and use the specialization at a Ω-point of U, as in 2.1.2.1.

2.1.3. *The lattice* $\Lambda_{\mathbb{Z}_\ell}$. For every $s\in S$, define

$$\Lambda_{\mathbb{Z}_\ell,s} := \mathrm{im}[\mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Z}_\ell} \to \mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Z}_\ell} \stackrel{c_{\ell,s}}{\to} V^{\mathrm{free}}_{\mathbb{Z}_\ell}] \subset V^{\mathrm{free}}_{\mathbb{Z}_\ell}.$$

By construction and 2.1.2, one has

$$\Lambda_{\mathbb{Z}_\ell,s} \subset V^{\mathrm{free},a}_{\mathbb{Z}_\ell,\eta} \subset V^{\mathrm{free},a}_{\mathbb{Z}_\ell,s} \subset V^{\mathrm{free}}_{\mathbb{Z}_\ell}.$$

Lemma 5. *The lattice $\Lambda_{\mathbb{Z}_\ell} := \Lambda_{\mathbb{Z}_\ell,s} \subset V^{\mathrm{free}}_{\mathbb{Z}_\ell}$ is independent of s (modulo the identifications $V_{\mathbb{Z}_\ell} = \mathcal{V}_{\mathbb{Z}_\ell,\bar{s}} \simeq \mathcal{V}_{\mathbb{Z}_\ell,\bar{\eta}}$).*

Proof. This follows from the fact that the restriction morphism $\mathrm{H}^{2i}(X_{\bar{k}},\mathbb{Z}_\ell(i)) \to \mathrm{H}^{2i}(X_{\bar{s}},\mathbb{Z}_\ell(i)) = V_{\mathbb{Z}_\ell}$ factors through the edge morphism $\epsilon : \mathrm{H}^{2i}(X_{\bar{k}},\mathbb{Z}_\ell(i)) \twoheadrightarrow E^{0,i}_\infty \hookrightarrow E^{0,i}_2 = \mathrm{H}^0(S_\infty, R^{2i}f_*\mathbb{Z}_\ell(i))$ of the Leray spectral sequence for $f: X\to S$ as

$$\begin{tikzcd}
\mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Z}_\ell} \arrow[rr, "|_{X_{\bar{s}}}"] \arrow[d, "c_\ell"'] & & \mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Z}_\ell} \arrow[d, "c_{\ell,s}"] \\
\mathrm{H}^{2i}(X_{\bar{k}},\mathbb{Z}_\ell(i)) \arrow[r, "\epsilon"] & \mathrm{H}^0(S_{\bar{k}}, R^{2i}f_*\mathbb{Z}_\ell(i)) \arrow[r, "(-)_{\bar{s}}"] & V^{\mathrm{free}}_{\mathbb{Z}_\ell}
\end{tikzcd}$$

and the fact the embedding

$$V^{\mathrm{free}}_{\mathbb{Z}_\ell} \cap (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell} = \mathrm{im}[\mathrm{H}^0(S_{\bar{k}}, R^{2i}f_*\mathbb{Z}_\ell(i)) \stackrel{(-)_{\bar{s}}}{\to} V^{\mathrm{free}}_{\mathbb{Z}_\ell}] \subset V^{\mathrm{free}}_{\mathbb{Z}_\ell}$$

is independent of s (*modulo* the identifications $V_{\mathbb{Z}_\ell} = \mathcal{V}_{\mathbb{Z}_\ell,\bar{s}} \simeq \mathcal{V}_{\mathbb{Z}_\ell,\bar{\eta}}$). □

Remark 6. Assume[2] there exists a smooth compactification $X \hookrightarrow X^{\mathrm{cpt}}$. Then the surjectivity of the restriction morphism $\mathrm{CH}^i(X^{\mathrm{cpt}}_{\bar{k}}) \twoheadrightarrow \mathrm{CH}^i(X_{\bar{k}})$ and the functoriality of cycle class maps shows that $\Lambda_{\mathbb{Z}_\ell}$ can also be described as

$$\Lambda_{\mathbb{Z}_\ell} = \mathrm{im}[\mathrm{CH}^i(X^{\mathrm{cpt}}_{\bar{k}})_{\mathbb{Z}_\ell} \stackrel{c_\ell}{\to} \mathrm{H}^{2i}(X^{\mathrm{cpt}}_{\bar{k}},\mathbb{Z}_\ell(i)) \to \mathrm{H}^{2i}(X^{\mathrm{cpt}}_{\bar{s}},\mathbb{Z}_\ell(i)) \twoheadrightarrow V^{\mathrm{free}}_{\mathbb{Z}_\ell}].$$

In particular, if $\bar{k}\hookrightarrow \Omega$ is an extension of algebraically closed fields and s_Ω a geometric point on S_Ω over $\bar{s}$, then 2.1.2.2 shows that

$$\Lambda_{\mathbb{Z}_\ell} = \mathrm{im}[\mathrm{CH}^i(X_\Omega)_{\mathbb{Z}_\ell} \to \mathrm{CH}^i(X_{s_\Omega})_{\mathbb{Z}_\ell} \stackrel{c_{\ell,s_\Omega}}{\to} V^{\mathrm{free}}_{\mathbb{Z}_\ell}].$$

2.2. **Strategy for the proof of Theorem A and Theorem B.** We retain the notation and conventions of Subsection 1.2 and Subsection 2.1.1. For every $s\in S$, set

$$\mathrm{Ob}^{\mathrm{free}}_{\mathbb{Z}_\ell,s} := |(C^{\mathrm{free}}_{\mathbb{Z}_\ell,s})_{\mathrm{tors}}|.$$

As

$$\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell,s} \le |(V_{\mathbb{Z}_\ell})_{\mathrm{tors}}|\mathrm{Ob}^{\mathrm{free}}_{\mathbb{Z}_\ell,s}$$

and as $(V_{\mathbb{Z}_\ell})_{\mathrm{tors}}$ is independent of $s\in S$ and, if[3] $p=0$, $(V_{\mathbb{Z}_\ell})_{\mathrm{tors}}=0$, $\ell\gg 0$, it is enough to prove Theorem A, Theorem B for $\mathrm{Ob}^{\mathrm{free}}_{\mathbb{Z}_\ell,s}$ instead of $\widetilde{\mathrm{Ob}}_{\mathbb{Z}_\ell,s}$.

2.2.1. *$\mathcal{V}_{\mathbb{Q}_\ell}$-generic points.* The proofs of Theorem A and Theorem B are parallel and follow from the combination of two independent statements involving $\mathcal{V}_{\mathbb{Q}_\ell}$-generic points. Let $\mathcal{V}_{\mathbb{Z}_\ell}$ be a $\mathbb{Z}_\ell$-local system on S.

2.2.1.1. *$\mathcal{V}_{\mathbb{Q}_\ell}$-generic points.* Define the sets of closed $\mathcal{V}_{\mathbb{Q}_\ell}$-generic points to be the subset $|S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}} \subset |S|$ of all $s\in|S|$ satisfying the following equivalent conditions

$$G^\circ_{\ell,s} = G^\circ_\ell \Leftrightarrow G^\circ_{\ell,s} \supset G^\circ_\ell \Leftrightarrow G^\circ_{\ell,s} \supset \overline{G}^\circ_\ell,$$

and let $|S|^{\mathrm{ngen}}_{\mathcal{V}_{\mathbb{Q}_\ell}} := |S| \setminus |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}} \subset |S|$ be the subset of closed non-$\mathcal{V}_{\mathbb{Q}_\ell}$-generic points. Note that $|S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$ is contained in the set of all $s\in|S|$ such that $V^a_{\mathbb{Q}_\ell,s} \subset (V_{\mathbb{Q}_\ell})^{\overline{G}^\circ_\ell}$.

[2] If $p=0$, this is always the case - see [Na62], [Na63], [Hi64].

[3] This follows from Artin's comparison - see Subsection 3.1.2 and the fact that singular cohomology groups are finitely generated. This is also true if $p>0$ [G83] but we will not resort to this fact.

2.2.1.2. *Sparcity.* Under mild assumptions one expects non-$\mathcal{V}_{\mathbb{Q}_\ell}$-generic points to be sparce - see [C23] for details. When S is a curve, one has the following unconditional results. Let $\overline{\Pi}_\ell$ denote the image of $\pi_1(S_{\bar{k}})$ acting on $V_{\mathbb{Q}_\ell}$ and, if $p > 0$, let $\overline{\Pi}_\ell^+ (\supset \overline{\Pi}_\ell)$ denote the image of $\pi_1(S_{k\bar{\mathbb{F}}_p})$ acting on $V_{\mathbb{Q}_\ell}$; these are ℓ-adic Lie groups. One says that $\mathcal{V}_{\mathbb{Q}_\ell}$ is:

- GLP (geometrically Lie perfect) if $\mathrm{Lie}(\overline{\Pi}_\ell)$ is a perfect Lie algebra *viz* one has $[\mathrm{Lie}(\overline{\Pi}_\ell), \mathrm{Lie}(\overline{\Pi}_\ell)] = 0$;
- and, if $p > 0$, GLU (geometrically Lie unrelated) if $\mathrm{Lie}(\overline{\Pi}_\ell)$ and $\mathrm{Lie}(\overline{\Pi}_\ell^+)$ have no non-trivial common quotient.

Fact A. ([CT13, Thm. 1]). *Assume $p = 0$, S is a curve and $\mathcal{V}_{\mathbb{Q}_\ell}$ is GLP. Then for every integer $d \geq 1$, the set $|S|^{\mathrm{ngen}}_{\mathcal{V}_{\mathbb{Q}_\ell}} \cap |S|^{\leq d}$ is finite.*

Fact B. ([T24]; see also the discussion in [A23, 1.7.1]). *Assume $p > 0$, S is a curve and $\mathcal{V}_{\mathbb{Q}_\ell}$ is GLU. Then the set $|S|^{\mathrm{ngen}}_{\mathcal{V}_{\mathbb{Q}_\ell}} \cap S(k)$ is finite.*

The $\mathbb{Z}_\ell$-local system $\mathcal{V}_\ell = R^{2i} f_* \mathbb{Q}_\ell(i)$ is GLP [D71], [D80]. If $p > 0$, it is not necessarily GLU but still, it is *e.g.* if $\overline{\Pi}_\ell$ is open in the derived subgroup of the image of $\pi_1(S_{\bar{k}})$ acting on $V_{\mathbb{Q}_\ell}$ - see [A23, Rem. 1.7.1.4] for details.

2.2.2. *The main Lemmas.* Fact A immediately reduce the proof of Theorem A to the proof of:

Lemma A. *Set $\mathcal{V}_{\mathbb{Z}_\ell} := R^{2i} f_* \mathbb{Z}_\ell(i)$. Assume $p = 0$ and* $\mathrm{VSing}(f_\infty, i)$ *holds for some (equivalently every) embedding $\infty : k \hookrightarrow \mathbb{C}$. Then,*

$$\mathrm{Ob}^{\mathrm{free,gen}}_{\mathbb{Z}_\ell} := \sup\{\mathrm{Ob}^{\mathrm{free}}_{\mathbb{Z}_\ell, s} \mid s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}\} < +\infty,$$

and $\mathrm{Ob}^{\mathrm{free,gen}}_{\mathbb{Z}_\ell} = 1$ *for* $\ell \gg 0$.

The proof of Lemma A will be carried out in Section 3.1.4.

Similarly, Fact B immediately reduces the proof of Theorem B to the proof of:

Lemma B. *Set $\mathcal{V}_{\mathbb{Z}_\ell} := R^{2i} f_* \mathbb{Z}_\ell(i)$. Assume $p > 0$ and either (i)* $\mathrm{WVEt}_{\mathbb{Q}_\ell}(f, i)$ *or (ii)* $\mathrm{VCrys}(f, i) + \mathrm{CrysEt}_{\mathbb{Q}_\ell}(f, i)$ *holds. Then,* $\mathrm{Ob}^{\mathrm{free,gen}}_{\mathbb{Z}_\ell} < +\infty$.

The proof of Lemma Lemma B will be carried out in Section 3.2.2.

Note that Lemma A and Lemma B do not involve any restriction on the dimension of S nor on the degree of the residue field $k(s)$ for $s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$.

Remark 7. *A priori*, the assumptions in Lemma A, Lemma B do not imply $\mathrm{Tate}_{\mathbb{Q}_\ell}(X_s, i)$, $s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$. However, if one assumes $\mathrm{Tate}_{\mathbb{Q}_\ell}(X_{s_0}, i)$ holds for some $s_0 \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$ then these assumptions indeed imply $\mathrm{Tate}_{\mathbb{Q}_\ell}(X_s, i)$, $s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$. Indeed, the proofs of Lemma A, Lemma B will show these assumptions imply $\Lambda_{\mathbb{Q}_\ell} = V^a_{\mathbb{Q}_\ell, s}$, $s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$, where $\Lambda_{\mathbb{Q}_\ell} = \Lambda_{\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Assume furthermore $\mathrm{Tate}_{\mathbb{Q}_\ell}(X_{s_0}, i)$ holds - that is $V^a_{\mathbb{Q}_\ell, s_0} = \widetilde{V}_{\mathbb{Q}_\ell, s_0}$, for some $s_0 \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$. But then, for every $s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$, one has

$$V^a_{\mathbb{Q}_\ell, s} = \Lambda_{\mathbb{Q}_\ell} = V^a_{\mathbb{Q}_\ell, s_0} = \widetilde{V}_{\mathbb{Q}_\ell, s_0} \overset{(\alpha)}{=} \widetilde{V}_{\mathbb{Q}_\ell, s},$$

where (α) follows from $s_0 \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$.

2.2.3. *Reduction to connected monodromy groups.* To bound $\mathrm{Ob}^{\mathrm{free}}_{\mathbb{Z}_\ell, s}$ uniformly for $s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$, one can freely replace $f : X \to S$ by a base change along a finite cover $\pi : S' \to S$ of connected smooth varieties over k. Indeed, consider the base-change diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r] & S \end{tikzcd}$$

and write $\mathcal{V}'_{\mathbb{Z}_\ell} := R^{2i}f'_*\mathbb{Z}_\ell(i)$. For $s \in |S|$ and $s' \in |S'|$ over $s \in |S|$, let $\bar{s}'$ be a geometric point over s' and let $\bar{s} = \pi \circ \bar{s}'$ denote its image on S. Then, $X'_{\bar{s}'} \tilde{\to} X_{\bar{s}}$ as $\bar{k}$-schemes hence, *a fortiori*, $\mathrm{CH}^i(X'_{\bar{s}'}) \tilde{\to} \mathrm{CH}^i(X_{\bar{s}})$. On the other hand, by proper base change, $\mathcal{V}'_{\mathbb{Z}_\ell} = \pi^*\mathcal{V}_{\mathbb{Z}_\ell}$ hence, one gets a canonical commutative square

$$\begin{tikzcd} \mathrm{CH}^i(X_{\bar{s}}) \arrow[r, "c_{\ell,s}"] & \mathrm{H}^{2i}(X_{\bar{s}}, \mathbb{Z}_\ell(i)) \arrow[d, equal] \\ \mathrm{CH}^i(X'_{\bar{s}'}) \arrow[u, "\simeq"] \arrow[r, "c_{\ell,s'}"] & \mathrm{H}^{2i}(X'_{\bar{s}'}, \mathbb{Z}_\ell(i)) \end{tikzcd},$$

where the vertical arrows are isomorphisms and the right vertical one is equivariant with respect to the functorial morphism $\pi_1(S') \hookrightarrow \pi_1(S)$. In particular, as $\pi_1(S') \hookrightarrow \pi_1(S)$ is open, one has $s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$ if and only if $s' \in |S'|^{\mathrm{gen}}_{\mathcal{V}'_{\mathbb{Q}_\ell}}$.

After base change along a finite cover $S' \to S$ of smooth varieties (which, working componentwise, we may assume to be connected and, replacing k by a finite field extension, geometrically connected over k), one may assume $\mathrm{VSing}^0(f'_\infty, i)$ (resp. $\mathrm{WVEt}^0_{\mathbb{Q}_\ell}(f', i)$, resp. $\mathrm{VCrys}^0(f', i)$) holds for every base change along a finite cover $S'_\infty \to S_\infty$ (resp. $S' \to S$, resp. $\mathcal{S}' \to \mathcal{S}$) of smooth varieties. Then, the assumptions and conclusions of Theorem A and Theorem B become unchanged by base change along finite covers of smooth varieties, so that one may assume:

a) the algebraic group $\overline{G}_\ell$ is connected[4];

b) the algebraic groups $G_{\ell,s}$, $s \in S$ are all connected[5].

2.2.4. *An elementary lemma.* Recall that for every $s \in S$, we identify $V_{\mathbb{Z}_\ell} := \mathcal{V}_{\mathbb{Z}_\ell,\bar{s}} \tilde{\to} \mathcal{V}_{\mathbb{Z}_\ell,\bar{\eta}}$. For a subset $\Sigma \subset S$, set

$$V^{\mathrm{free},a}_{\mathbb{Z}_\ell,\Sigma} := \bigcap_{s\in\Sigma} V^{\mathrm{free},a}_{\mathbb{Z}_\ell,s} \subset V^{\mathrm{free},a}_{\mathbb{Z}_\ell,s} \subset V^{\mathrm{free}}_{\mathbb{Z}_\ell}.$$

Lemma 8. *For every $\mathbb{Z}_\ell$-submodule $T_{\mathbb{Z}_\ell} \subset V^{\mathrm{free},a}_{\mathbb{Z}_\ell,\Sigma}$ and for every $s \in \Sigma$, one has the following implications*

$$T_{\mathbb{Q}_\ell} = V^a_{\mathbb{Q}_\ell,s} \Longleftrightarrow [V^{\mathrm{free},a}_{\mathbb{Z}_\ell,s} : T_{\mathbb{Z}_\ell}] < +\infty \Longrightarrow \mathrm{Ob}^{\mathrm{free}}_{\mathbb{Z}_\ell,s} \leq c(T_{\mathbb{Z}_\ell}) := |(V^{\mathrm{free}}_{\mathbb{Z}_\ell}/T_{\mathbb{Z}_\ell})_{\mathrm{tors}}|.$$

Proof. The first equivalence is straightforward. The second implication follows from the canonical commutative diagram of short exact sequences

(4)
$$\begin{tikzcd} 0 \arrow[r] & T_{\mathbb{Z}_\ell} \arrow[r] \arrow[d, hook] & V^{\mathrm{free}}_{\mathbb{Z}_\ell} \arrow[r] \arrow[d, equal] & V^{\mathrm{free}}_{\mathbb{Z}_\ell}/T_{\mathbb{Z}_\ell} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V^{\mathrm{free},a}_{\mathbb{Z}_\ell,s} \arrow[r] & V^{\mathrm{free}}_{\mathbb{Z}_\ell} \arrow[r] & C^{\mathrm{free}}_{\mathbb{Z}_\ell,s} \arrow[r] & 0 \end{tikzcd}$$

which, by the snake lemma, identifies

$$Q_{\mathbb{Z}_\ell,s} := \mathrm{coker}[T_{\mathbb{Z}_\ell} \hookrightarrow V^{\mathrm{free},a}_{\mathbb{Z}_\ell,s}] \tilde{\to} \ker[V^{\mathrm{free}}_{\mathbb{Z}_\ell}/T_{\mathbb{Z}_\ell} \twoheadrightarrow C^{\mathrm{free}}_{\mathbb{Z}_\ell,s}] =: K_{\mathbb{Z}_\ell,s}.$$

But if $K_{\mathbb{Z}_\ell,s}$ is finite, one gets a short exact sequence

$$0 \to K_{\mathbb{Z}_\ell,s} \to (V^{\mathrm{free}}_{\mathbb{Z}_\ell}/T_{\mathbb{Z}_\ell})_{\mathrm{tors}} \to (C^{\mathrm{free}}_{\mathbb{Z}_\ell,s})_{\mathrm{tors}} \to 0,$$

whence the assertion. □

[4] First, after replacing k by a finite field extension, one may assume $S(k) \neq \emptyset$, so that fixing $s \in S(k)$ yields a splitting $s : \pi_1(s) = \pi_1(k) \hookrightarrow \pi_1(S)$ of the canonical short exact sequence

$$1 \to \pi_1(S_{\bar{k}}) \to \pi_1(S) \to \pi_1(k) \to 1$$

and a well-defined action by conjugacy of $\pi_1(k)$ on $\pi_1(S)$. Then, let $S'_{\bar{k}} \to S_{\bar{k}}$ denote the connected étale cover corresponding to $\ker(\pi_1(S_{\bar{k}}) \to \pi_0(\overline{G}_\ell))$. As $\overline{G}^\circ_\ell$ is normal in G_ℓ, the $\pi_1(k)$-action stabilizes $\pi_1(S'_{\bar{k}})$ hence $s(\pi_1(k))\pi_1(S'_{\bar{k}}) \subset \pi_1(S)$ is an open subgroup corresponding to a connected étale cover $S' \to S$ which, by construction, has the requested property.

[5] After base-change along the connected étale cover $S' \to S$ trivializing $\mathcal{V}_\ell/\tilde{\ell}$ (with $\tilde{\ell} = 4$ if $\ell = 2$ and $\tilde{\ell} = \ell$ if $\ell \neq 2$, this classically follows from the Cebotarev density theorem, using Frobenius tori.

Lemma 8 reduces the proof of Lemma A and Lemma B to finding a $\mathbb{Z}_\ell$-submodule $T_{\mathbb{Z}_\ell} \subset V^{\mathrm{free},a}_{\mathbb{Z}_\ell,\Sigma}$ such that $T_{\mathbb{Q}_\ell} = V^a_{\mathbb{Q}_\ell,s}$, $s \in \Sigma = |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$ and, in the setting of of Lemma A, such that $c(T_{\mathbb{Z}_\ell}) = 0$, $\ell \gg 0$. In all cases, we will consider the $\mathbb{Z}_\ell$-submodule $T_{\mathbb{Z}_\ell} := \Lambda_{\mathbb{Z}_\ell}$ introduced in Subsection 2.1.3, Lemma 5. As a warm-up, we end this Section with the proof of Lemma B (i).

2.2.5. *Proof of Lemma B (i).* Let $s \in \Sigma = |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$. Assuming $\mathrm{WVEt}_{\mathbb{Q}_\ell}(f,i)$, we are to prove that the inclusion $\Lambda_{\mathbb{Q}_\ell} \subset V^a_{\mathbb{Q}_\ell,s}$ is an equality. This follows from the inclusions

$$V^a_{\mathbb{Q}_\ell,s} = V^a_{\mathbb{Q}_\ell,s} \cap \widetilde{V}_{\mathbb{Q}_\ell,s} \overset{(\alpha)}{=} V^a_{\mathbb{Q}_\ell,s} \cap \widetilde{V}_{\mathbb{Q}_\ell,\eta} \overset{(\beta)}{\subset} V^a_{\mathbb{Q}_\ell,s} \cap (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell} \overset{(\gamma)}{=} \Lambda_{\mathbb{Q}_\ell} \subset V_{\mathbb{Q}_\ell},$$

where (α) follows from $s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$, (β) from the reduction 2.2.3 a), and (γ) is $\mathrm{WVEt}_{\mathbb{Q}_\ell}(f,i)$.

3. Comparison with singular and crystalline cohomologies

3.1. Singular cohomology.

3.1.1. *Singular $\mathbb{Z}$-local systems.* Let S_∞ be a connected variety smooth over $\mathbb{C}$. For every $s_{0\infty}, s_\infty \in S_\infty(\mathbb{C}) = S^{\mathrm{an}}_\infty$, fix a topological path $s_\infty \to s_{0\infty}$, inducing an isomorphism of fiber functors $\alpha_{s_\infty} : (-)_{s_\infty} \tilde{\to} (-)_{s_{0\infty}}$. In particular, for every singular $\mathbb{Z}$-local system $\mathcal{V}_\mathbb{Z}$ on S^{an}_∞, one identifies $\mathcal{V}_{\infty,\mathbb{Z},s_\infty} \tilde{\to} \mathcal{V}_{\infty,\mathbb{Z},s_{0\infty}}$ equivariantly with respect to the isomorphism of topological fundamental groups $\pi_1^{\mathrm{top}}(S^{\mathrm{an}}_\infty, s_\infty) \tilde{\to} \pi_1^{\mathrm{top}}(S^{\mathrm{an}}_\infty, s_{0\infty})$, $\gamma \mapsto \alpha_{s_\infty}\gamma\alpha_{s_\infty}^{-1}$. So that we will in general omit fiber functors from our notation and simply write

$$V_\mathbb{Z} := \mathcal{V}_{\mathbb{Z},s_\infty} \tilde{\to} \mathcal{V}_{\mathbb{Z},s_{0\infty}}.$$

Let $f_\infty : X_\infty \to S_\infty$ be a smooth projective morphism. The singular $\mathbb{Z}$-local system $\mathcal{V}_\mathbb{Z} := R^{2i} f^{\mathrm{an}}_\infty \mathbb{Z}(i)$ on S^{an}_∞ underlies a polarizable $\mathbb{Z}$-variation of Hodge structure. Let $G \subset \mathrm{GL}(V_\mathbb{Q})$ denote the generic Mumford-Tate group of $\mathcal{V}_\mathbb{Q} := \mathcal{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$, and for every $s_\infty \in S_\infty(\mathbb{C})$, let $G_{s_\infty} \subset G$ denote the Mumford-Tate group of the polarizable $\mathbb{Q}$-Hodge structure $s^*_\infty \mathcal{V}_\mathbb{Q}$. Let also $\overline{G} \subset \mathrm{GL}(V_\mathbb{Q})$ denote the Zariski-closure of the image of $\pi_1^{\mathrm{top}}(S^{\mathrm{an}}_\infty)$ acting on $V_\mathbb{Q}$. By the fixed part theorem, $\overline{G}^\circ$ a normal closed subgroup of G and, for every $s_\infty \in S_\infty(\mathbb{C})$, one has $G = \overline{G}^\circ G_{s_\infty}$.

As in Subsection 2.1.3, for every $s_\infty \in S_\infty(\mathbb{C})$ set

$$\Lambda_{\mathbb{Z},s_\infty} := \mathrm{im}[\mathrm{CH}^i(X_\infty) \to \mathrm{CH}^i(X_{s_\infty}) \overset{c_{s_\infty}}{\to} V^{\mathrm{free}}_\mathbb{Z}] \subset V^{\mathrm{free}}_\mathbb{Z}.$$

The same argument as in the proof of Lemma 5 (using Leray spectral sequence for singular cohomology) shows that $\Lambda_\mathbb{Z} := \Lambda_{\mathbb{Z},s_\infty}$ is independent of $s_\infty \in S_\infty(\mathbb{C})$.

3.1.2. *Artin's comparison.* Assume $p = 0$ and fix an embedding $\infty : k \hookrightarrow \mathbb{C}$. Recall that $(-)_\infty$ denotes the base-change functor along $\mathrm{Spec}(\mathbb{C}) \overset{\infty}{\to} \mathrm{Spec}(k)$ and $(-)^{\mathrm{an}}$ the analytification functor from varieties over $\mathbb{C}$ to complex analytic spaces. Let S be a geometrically connected, smooth variety over k. For every $s_\infty \in S_\infty(\mathbb{C})$ over $s \in S$ let $k(\bar{s}) \subset \mathbb{C}$ denote the algebraic closure of $k(s)$ determined by $k(s) \hookrightarrow \mathbb{C}$ and let $\bar{s}$ denote the corresponding geometric point over s. Let $f : X \to S$ be a smooth projective morphism. The local systems $\mathcal{V}_\mathbb{Z} := R^{2i} f^{\mathrm{an}}_\infty \mathbb{Z}(i)$ on S^{an}_∞ and $\mathcal{V}_{\mathbb{Z}_\ell} := R^{2i} f^{\mathrm{an}}_\infty \mathbb{Z}_\ell(i)$ on S are related by Artin's comparison isomorphism [SGA4, XI]

$$\mathcal{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_\ell \tilde{\to} \mathcal{V}^{\mathrm{an}}_{\mathbb{Z}_\ell}, \tag{5}$$

where we write $\mathcal{V}^{\mathrm{an}}_{\mathbb{Z}_\ell}$ for the pull-back of $\mathcal{V}_{\mathbb{Z}_\ell}$ along[6] the morphisms of sites $(X^{\mathrm{an}}_\infty)_{\mathrm{an}} \to X_{\infty,\mathrm{et}} \to X_{\mathrm{et}}$. Equivalently, for every $s_\infty \in S_\infty(\mathbb{C})$ over $s \in |S|$, one has a canonical isomorphism of $\mathbb{Z}_\ell$-modules

$$V_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}_\ell = \mathcal{V}_{\mathbb{Z},s_\infty} \otimes_\mathbb{Z} \mathbb{Z}_\ell \tilde{\to} \mathcal{V}_{\mathbb{Z}_\ell,\bar{s}} = V_{\mathbb{Z}_\ell}, \quad V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{Q}_\ell \tilde{\to} V_{\mathbb{Q}_\ell}, \tag{6}$$

which is equivariant with respect to the profinite completion morphism composed with the GAGA isomorphism and the projection

$$\pi_1^{\mathrm{top}}(S^{\mathrm{an}}_\infty) \to \pi_1^{\mathrm{top}}(S^{\mathrm{an}}_\infty)^\wedge \tilde{\to} \pi_1(S_\infty) \tilde{\to} \pi_1(S_{\bar{k}}) \hookrightarrow \pi_1(S).$$

In particular, $\overline{G} \subset \mathrm{GL}(V_\mathbb{Q})$ identifies, *modulo* (6), with the scalar extension $\overline{G}_{\mathbb{Q}_\ell} \subset \mathrm{GL}(V_\mathbb{Q} \otimes_\mathbb{Q} \mathbb{Q}_\ell)$ of $\overline{G} \subset \mathrm{GL}(V_\mathbb{Q})$.

[6] More precisely, write $\mathcal{V}_{\mathbb{Z}_\ell} = \lim_n \mathcal{V}_{\mathbb{Z}/\ell^n}$ as a limit of $\mathbb{Z}/\ell^n$-local systems and define the analytification of $\mathcal{V}_{\mathbb{Z}_\ell}$ as $(\mathcal{V}_{\mathbb{Z}_\ell})^{\mathrm{an}} := \lim_n \mathcal{V}_{\mathbb{Z}/\ell^n}|_{(X^{\mathrm{an}}_\infty)_{\mathrm{an}}}$.

Artin's comparison isomorphism is compatible with cycle class maps on both sides. Namely, for every $s_\infty \in S_\infty(\mathbb{C})$ over $s \in S$ one has a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^i(X_{\bar{k}}) \arrow[r, "|_{X_{\bar{s}}}"] \arrow[d, "|_{X_\infty}"'] & \mathrm{CH}^i(X_{\bar{s}}) \arrow[r, "c_{\ell,s}"] \arrow[d, "|_{X_{s_\infty}}"'] & V_{\mathbb{Z}_\ell}^{\mathrm{free}} & \\
\mathrm{CH}^i(X_\infty) \arrow[r, "|_{X_{s_\infty}}"'] & \mathrm{CH}^i(X_{s_\infty}) \arrow[r, "c_{s_\infty}"'] & V_{\mathbb{Z}}^{\mathrm{free}} \arrow[u, hook] \arrow[r, hook, "-\otimes_{\mathbb{Z}}\mathbb{Z}_\ell"'] & V_{\mathbb{Z}}^{\mathrm{free}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell. \arrow[ul, "(6)"', "\simeq"]
\end{tikzcd}$$

As a result, we will identify subgroups of $V_{\mathbb{Z}}^{\mathrm{free}}$ (*e.g.* $\Lambda_{\mathbb{Z}}$, $V_{\mathbb{Z},s_\infty}^{\mathrm{free},a}$ *etc.*) with their image in $V_{\mathbb{Z}_\ell}^{\mathrm{free}}$. Set

$$\Lambda_{\ell,\mathbb{Z}} := \mathrm{im}[\mathrm{CH}^i(X_{\bar{k}}) \to \mathrm{CH}^i(X_{\bar{s}}) \stackrel{c_{\ell,s}}{\to} V_{\mathbb{Z}_\ell}^{\mathrm{free}}] \subset V_{\ell,\mathbb{Z},s}^{\mathrm{free},a} := \mathrm{im}[\mathrm{CH}^i(X_{\bar{s}}) \stackrel{c_{\ell,s}}{\to} V_{\mathbb{Z}_\ell}^{\mathrm{free}}].$$

Then, from 2.1.2.2 and Remark 6 applied to $\bar{k} \hookrightarrow \mathbb{C}$, one has

$$\Lambda_{\mathbb{Z}} = \Lambda_{\ell,\mathbb{Z}}, \quad V_{\mathbb{Z},s_\infty}^{\mathrm{free},a} = V_{\ell,\mathbb{Z},s}^{\mathrm{free},a},$$

hence

$$\Lambda_{\ell,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \tilde{\to} \Lambda_{\mathbb{Z}_\ell}, \quad V_{\ell,\mathbb{Z},s}^{\mathrm{free},a} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \tilde{\to} V_{\mathbb{Z}_\ell,s}^{\mathrm{free},a}. \tag{7}$$

3.1.3. *Proof of Proposition 2.* For every $s \in S$, write

$$\Lambda_{\ell,\mathbb{Q}} = \mathrm{im}[\mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Q}} \to \mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Q}} \stackrel{c_{\ell,s}}{\to} V_{\mathbb{Q}_\ell}] \subset V_{\ell,\mathbb{Q},s}^a := \mathrm{im}[\mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Q}} \stackrel{c_{\ell,s}}{\to} V_{\mathbb{Q}_\ell}] \subset V_{\mathbb{Q}_\ell,s}^a,$$

$$\Lambda_{\mathbb{Q}_\ell} = \mathrm{im}[\mathrm{CH}^i(X_{\bar{k}})_{\mathbb{Q}_\ell} \to \mathrm{CH}^i(X_{\bar{s}})_{\mathbb{Q}_\ell} \stackrel{c_{\ell,s}}{\to} V_{\mathbb{Q}_\ell}].$$

If $p = 0$, fix an embedding $\infty : k \hookrightarrow \mathbb{C}$ and, for every $s_\infty \in S_\infty(\mathbb{C})$, write

$$\Lambda_{\mathbb{Q}} = \mathrm{im}[\mathrm{CH}^i(X_\infty)_{\mathbb{Q}} \to \mathrm{CH}^i(X_{s_\infty})_{\mathbb{Q}} \stackrel{c_{s_\infty}}{\to} V_{\mathbb{Q}}] \subset V_{\mathbb{Q},s_\infty}^a.$$

Recall from Subsection 3.1.1 and Subsection 2.1.3 that $\Lambda_{\mathbb{Q}}$ is independent of s_∞ and $\Lambda_{\ell,\mathbb{Q}}$, $\Lambda_{\mathbb{Q}_\ell}$ are independent of s (as the notation suggests) and, if $p = 0$, from Subsection 3.1.2, that $\Lambda_{\ell,\mathbb{Q}} = \Lambda_{\mathbb{Q}}$.

With these notation, $\mathrm{VSing}^0(f_\infty, i)$, $\mathrm{VEt}^0_{\mathbb{Q}_\ell}(f, i)$ and $\mathrm{WVEt}^0_{\mathbb{Q}_\ell}(f, i)$ can be reformulated as

$$\begin{array}{lll}
\mathrm{VSing}^0(f_\infty, i) & V_{\mathbb{Q},s_\infty}^a \cap (V_{\mathbb{Q}})^{\overline{G}} \subset \Lambda_{\mathbb{Q}}, & s_\infty \in S_\infty. \\
\mathrm{VEt}^0_{\mathbb{Q}_\ell}(f, i) & V_{\ell,\mathbb{Q},s}^a \cap (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell} \subset \Lambda_{\ell,\mathbb{Q}}, & s \in |S|. \\
\mathrm{WVEt}^0_{\mathbb{Q}_\ell}(f, i) & V_{\mathbb{Q}_\ell,s}^a \cap (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell} \subset \Lambda_{\mathbb{Q}_\ell}, & s \in |S|.
\end{array}$$

The implication $\mathrm{VEt}^0_{\mathbb{Q}_\ell}(f, i) \Rightarrow \mathrm{WVEt}^0_{\mathbb{Q}_\ell}(f, i)$ immediately follows from the fact that, for every $s \in S$, $V_{\mathbb{Q}_\ell,s}^a$ is the $\mathbb{Q}_\ell$-span of $V_{\ell,\mathbb{Q},s}^a$.

As $\mathrm{Tate}_{\mathbb{Q}_\ell}(X_\eta, i)$ is invariant under base-change along finite covers $S' \to S$ of smooth varieties, to prove $\mathrm{Tate}_{\mathbb{Q}_\ell}(X_\eta, i) \Rightarrow \mathrm{WVEt}_{\mathbb{Q}_\ell}(f, i)$ one may first perform such a base-change hence assume:

- $V_{\mathbb{Q}_\ell,\eta}^a = \mathrm{im}[\mathrm{CH}^i(X_\eta)_{\mathbb{Q}_\ell} \to \mathrm{CH}^i(X_{\bar{\eta}})_{\mathbb{Q}_\ell} \stackrel{c_{\ell,\eta}}{\to} V_{\mathbb{Q}_\ell}]$, which, from the surjectivity of the restriction map $\mathrm{CH}^i(X) \twoheadrightarrow \mathrm{CH}^i(X_\eta)$, implies $\Lambda_{\mathbb{Q}_\ell} = V_{\mathbb{Q}_\ell,\eta}^a$;

- $\overline{G}_\ell$ is connected - see Footnote 4, which ensures $V_{\mathbb{Q}_\ell,s}^a \cap (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell} \subset \widetilde{V}_{\mathbb{Q}_\ell,\eta} \stackrel{(\alpha)}{=} V_{\mathbb{Q}_\ell,\eta}^a = \Lambda_{\mathbb{Q}_\ell}$, where (α) is $\mathrm{Tate}_{\mathbb{Q}_\ell}(X_\eta, i)$.

If $p = 0$, for every $s_\infty \in S_\infty(\mathbb{C})$ above $s \in |S|$, Artin's comparison isomorphism yields the following canonical commutative diagram:

$$\begin{tikzcd}
V_{\mathbb{Q},s_\infty}^a \cap (V_{\mathbb{Q}})^{\overline{G}} \arrow[r, "\simeq"] \arrow[d, hook] & V_{\ell,\mathbb{Q},s}^a \cap (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell} \arrow[d, hook] \\
\Lambda_{\mathbb{Q}} \arrow[r, "\simeq"'] & \Lambda_{\ell,\mathbb{Q}},
\end{tikzcd} \tag{8}$$

which shows $\mathrm{VSing}^0(f_\infty, i) \Leftrightarrow \mathrm{VEt}^0_{\mathbb{Q}_\ell}(f, i)$, and the isomorphisms

$$(V^a_{\ell,\mathbb{Q},s} \cap (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = V^a_{\mathbb{Q}_\ell,s} \cap (V_{\mathbb{Q}_\ell})^{\overline{G}_\ell}, \ \ \Lambda_{\ell,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = \Lambda_{\mathbb{Q}_\ell},$$

(similar to (7)), which, together with (8), show $\mathrm{WVEt}^0_{\mathbb{Q}_\ell}(f, i) \Rightarrow \mathrm{VEt}^0_{\mathbb{Q}_\ell}(f, i)$.

3.1.4. *Proof of Lemma A.* As we already observed that $\mathrm{VSing}(f_\infty, i) \Leftrightarrow \mathrm{WVEt}_{\mathbb{Q}_\ell}(f, i)$ and $\mathrm{WVEt}_{\mathbb{Q}_\ell}(f, i) \Rightarrow \Lambda_{\mathbb{Q}_\ell} = V^a_{\mathbb{Q}_\ell,s}$, $s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\mathbb{Q}_\ell}}$ - see Subsection 2.2.5, it only remains to prove that $c(\Lambda_{\mathbb{Z}_\ell}) = 0$ for $\ell \gg 0$. This follows at once from Artin's comparison isomorphism, which yields the identifications

$$(V^{\mathrm{free}}_{\mathbb{Z}_\ell}/\Lambda_{\mathbb{Z}_\ell})_{\mathrm{tors}} \simeq (V^{\mathrm{free}}_{\mathbb{Z}}/\Lambda_{\mathbb{Z}})_{\mathrm{tors}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell.$$

and the fact that $(V^{\mathrm{free}}_{\mathbb{Z}}/\Lambda_{\mathbb{Z}})_{\mathrm{tors}}$ is a finite group.

3.1.5. *Obstruction to the integral Hodge conjecture.* In this subsection, we deduce from Artin's comparison and Theorem A uniform bounds for the obstruction to the integral Hodge conjecture.

Let X_∞ be a smooth, projective variety over $\mathbb{C}$. The cycle class map

$$c : \mathrm{CH}^i(X_\infty) \to V_{\mathbb{Z}} := \mathrm{H}^{2i}(X^{\mathrm{an}}_\infty, \mathbb{Z}(i))$$

for $\mathbb{Z}$-singular cohomology fits into a canonical diagram analogue to (1)

$$\begin{tikzcd}
\mathrm{CH}^i(X_\infty) \arrow[r, two heads] \arrow[rrr, bend left, "c"] \arrow[dd] & V^a_{\mathbb{Z}} \arrow[r, hook] \arrow[d, two heads] & \widetilde{V}_{\mathbb{Z}} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & V_{\mathbb{Z}} \arrow[d, two heads] \\
 & V^{\mathrm{free},a}_{\mathbb{Z}} \arrow[r, hook] \arrow[d, hook] & \widetilde{V}^{\mathrm{free}}_{\mathbb{Z}} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & V^{\mathrm{free}}_{\mathbb{Z}} \arrow[d, hook] \\
\mathrm{CH}^i(X_\infty)_{\mathbb{Q}} \arrow[r, two heads] & V^a_{\mathbb{Q}} \arrow[r, hook] & \widetilde{V}_{\mathbb{Q}} \arrow[r, hook] & V_{\mathbb{Q}},
\end{tikzcd}$$

where, writing $G \subset \mathrm{GL}(V_{\mathbb{Q}})$ for the Mumford-Tate group of the polarizable $\mathbb{Q}$-Hodge structure $V_{\mathbb{Q}}$ underlies,

$$\widetilde{V}_{\mathbb{Q}} := (V_{\mathbb{Q}})^G$$

is the $\mathbb{Q}$-vector space of Hodge classes. The (classical) rational $\mathbb{Q}$-Hodge conjecture in codimension i for X [H52]

$$\mathrm{Hodge}_{\mathbb{Q}}(X_\infty, i) \ \ V^a_{\mathbb{Q}} = \widetilde{V}_{\mathbb{Q}}$$

also admits integral variants:

$$\begin{array}{lll}
\mathrm{Hodge}^{\mathrm{free}}_{\mathbb{Z}}(X_\infty, i) & V^{\mathrm{free},a}_{\mathbb{Z}_\ell} = \widetilde{V}^{\mathrm{free}}_{\mathbb{Z}} & \text{(Integral Hodge conjecture modulo torsion);} \\
\mathrm{Hodge}_{\mathbb{Z}}(X_\infty, i) & V^a_{\mathbb{Z}} = \widetilde{V}_{\mathbb{Z}} & \text{(Integral Hodge conjecture).}
\end{array}$$

Again, the implications

$$\mathrm{Hodge}_{\mathbb{Z}}(X_\infty, i) \Rightarrow \mathrm{Hodge}^{\mathrm{free}}_{\mathbb{Z}}(X_\infty, i) \Rightarrow \mathrm{Hodge}_{\mathbb{Q}}(X_\infty, i)$$

are tautological and, in general, the converse implications are known to fail (see e.g. [AtH62, Ge19] for examples of the failure of $\mathrm{Hodge}_{\mathbb{Q}}(X_\infty, i)$ and [Ko90, K21] for examples of the failure of $\mathrm{Hodge}^{\mathrm{free}}_{\mathbb{Z}}(X_\infty, i)$). By definition, the obstructions to $\mathrm{Hodge}_{\mathbb{Q}}(X_\infty, i)$, $\mathrm{Hodge}^{\mathrm{free}}_{\mathbb{Z}}(X_\infty, i)$, $\mathrm{Hodge}_{\mathbb{Z}}(X_\infty, i)$ are, respectively:

$$\widetilde{C}_{\mathbb{Q}} := \widetilde{V}_{\mathbb{Q}}/V^a_{\mathbb{Q}}, \ \ \widetilde{C}^{\mathrm{free}}_{\mathbb{Z}} := \widetilde{V}^{\mathrm{free}}_{\mathbb{Z}}/V^{\mathrm{free},a}_{\mathbb{Z}}, \ \ \widetilde{C}_{\mathbb{Z}} := \widetilde{V}_{\mathbb{Z}}/V^a_{\mathbb{Z}},$$

with the properties that one has the short exact sequence

$$0 \to (V_{\mathbb{Z}})_{\mathrm{tors}}/(V^a_{\mathbb{Z}})_{\mathrm{tors}} \to \widetilde{C}_{\mathbb{Z}} \to \widetilde{C}^{\mathrm{free}}_{\mathbb{Z}} \to 0 \tag{9}$$

and that

$$\mathrm{Hodge}_{\mathbb{Q}} \Leftrightarrow (\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}})_{\mathrm{tors}} = \widetilde{C}^{\mathrm{free}}_{\mathbb{Z}} \Leftrightarrow (\widetilde{C}_{\mathbb{Z}})_{\mathrm{tors}} = \widetilde{C}_{\mathbb{Z}}$$

in which case, (9) reads

$$0 \to (V_{\mathbb{Z}})_{\mathrm{tors}}/(V^a_{\mathbb{Z}})_{\mathrm{tors}} \to (\widetilde{C}_{\mathbb{Z}})_{\mathrm{tors}} \to (\widetilde{C}^{\mathrm{free}}_{\mathbb{Z}})_{\mathrm{tors}} \to 0.$$

Furthermore,

$$(\widetilde{C}^{\text{free}}_{\mathbb{Z}})_{\text{tors}} = (C^{\text{free}}_{\mathbb{Z}})_{\text{tors}} := V^{\text{free}}_{\mathbb{Z}}/V^{\text{free},a}_{\mathbb{Z}}.$$

Assume $p = 0$ and fix an embedding $\infty : k \hookrightarrow \mathbb{C}$. Let X be a smooth projective variety over k. From the observations in Subsection 3.1.2 and the flatness of $\mathbb{Z} \hookrightarrow \mathbb{Z}_\ell$, Artin's comparison isomorphism induces the following identifications

$$((V_{\mathbb{Z}})_{\text{tors}}/(V^a_{\mathbb{Z}})_{\text{tors}}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \tilde{\to} (V_{\mathbb{Z}_\ell})_{\text{tors}}/(V^a_{\mathbb{Z}_\ell})_{\text{tors}}, \quad (C^{\text{free}}_{\mathbb{Z}})_{\text{tors}} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \tilde{\to} (C^{\text{free}}_{\mathbb{Z}_\ell})_{\text{tors}}.$$

As $V_{\mathbb{Z}}$ is a $\mathbb{Z}$-module of finite type, this shows, in particular,

a) $(\widetilde{C}^{\text{free}}_{\mathbb{Z}_\ell})_{\text{tors}} = 0$ - hence $(C^{\text{free}}_{\mathbb{Z}_\ell})_{\text{tors}} = 0$, for $\ell \gg 0$.

b) The obstruction $(C^{\text{free}}_{\mathbb{Z}})_{\text{tors}}$ to $\text{Hodge}^{\text{free}}_{\mathbb{Z}}(X_\infty, i)$ can be recovered from the obstructions $(C^{\text{free}}_{\mathbb{Z}_\ell})_{\text{tors}}$ to $\text{Tate}^{\text{free}}_{\mathbb{Z}_\ell}(X, i)$, when ℓ varies as

$$(C^{\text{free}}_{\mathbb{Z}})_{\text{tors}} = \oplus_\ell (C^{\text{free}}_{\mathbb{Z}_\ell})_{\text{tors}}.$$

As in Subsection 1.2, let now S be a smooth, geometrically connected variety over k and $f : X \to S$ a smooth projective morphism. For $s_\infty \in S_\infty(\mathbb{C})$ above $s \in S$, denote by a subscript $(-)_{s_\infty}$ the various modules attached to $X_{s_\infty} = X_{\infty,s_\infty}$ introduced above (*e.g.* $V_{\mathbb{Z},s_\infty} := \mathrm{H}^{2i}(X^{\text{an}}_{s_\infty}, \mathbb{Z}(i))$, $V^a_{\mathbb{Z},s_\infty} := \mathrm{im}[\mathrm{CH}^i(X_{s_\infty}) \to V_{\mathbb{Z}}]$ *etc.*). Again, one may investigate how

$$\widetilde{\mathrm{Ob}}_{\mathbb{Z},s} := |(\widetilde{C}_{\mathbb{Z},s_\infty})_{\text{tors}}|$$

vary with $s \in |S|$. A direct consequence of Theorem A and the observations a), b) above is the following.

Corollary 9. *Assume S is a curve and* $\mathrm{VSing}(f_\infty, i)$ *holds. Then, for every integer $d \geq 1$, one has*

$$\widetilde{\mathrm{Ob}}^{\leq d}_{\mathbb{Z}} := \sup\{\widetilde{\mathrm{Ob}}_{\mathbb{Z},s_\infty} \mid s \in |S|^{\leq d}\} < +\infty.$$

When $i = 2$, $(\widetilde{C}_{\mathbb{Z},s_\infty})_{\text{tors}}$ can again be described in terms of degree 3 unramified cohomology. More precisely, set $C_{\mathbb{Z},s_\infty} := V_{\mathbb{Z}_\ell}/V^a_{\mathbb{Z},s_\infty}$. From the short exact sequence

$$0 \to \widetilde{C}_{\mathbb{Z},s_\infty} \to C_{\mathbb{Z},s_\infty} \to V_{\mathbb{Z},s_\infty}/\widetilde{V}_{\mathbb{Z},s_\infty} \to 0$$

and the fact that $V_{\mathbb{Z},s_\infty}/\widetilde{V}_{\mathbb{Z},s_\infty}$ is torsion-free, one has $(\widetilde{C}_{\mathbb{Z},s_\infty})_{\text{tors}} = (C_{\mathbb{Z},s_\infty})_{\text{tors}}$. If $i = 2$, [CTV12, Thm. 3.7] establishes that $(C_{\mathbb{Z},s_\infty})_{\text{tors}}$ is isomorphic to

$$\mathrm{H}^3_{\text{nr}}(X^{\text{an}}_{\infty,s_\infty}, \mathbb{Q}/\mathbb{Z}(2))_{\text{ndiv}} \stackrel{def}{=} \mathrm{coker}[\mathrm{H}^3_{nr}(X^{\text{an}}_{\infty,s_\infty}, \mathbb{Q}/\mathbb{Z}(2))_{\text{div}} \to \mathrm{H}^3_{\text{nr}}(X^{\text{an}}_{\infty,s_\infty}, \mathbb{Q}/\mathbb{Z}(2))].$$

Hence Corollary 9 implies (see also [CTV12, Sec. 5.1]):

Corollary 10. *Assume S is a curve and* $\mathrm{VSing}(f_\infty, i)$ *holds. Then, for every integer $d \geq 1$,*

$$\sup\{|\mathrm{H}^3_{\text{nr}}(X^{\text{an}}_{\infty,s_\infty} \mathbb{Q}/\mathbb{Z}(2))_{\text{ndiv}}| \mid s \in |S|^{\leq d}\}| < +\infty.$$

Remark 11. a) Using [CTV12, Thm. 3.11] and Corollary 9 for cycles of dimension 1, one has an analogue of Corollary 10 with uniform bounds for the groups $\mathrm{H}^{n-3}(X^{\text{an}}_{\infty,s_\infty}, \mathcal{H}^n_{X^{\text{an}}_{\infty,s_\infty}}(\mathbb{Q}/\mathbb{Z}(n-1)))_{\text{ndiv}}$, where n is the relative dimension of $f : Y \to X$.

b) More generally, Corollary 10 holds with $\mathrm{H}^3_{nr}(X_{\infty,s}, \mathbb{Q}/\mathbb{Z}(2))_{\text{ndiv}}$ replaced by Schreieder's refined unramified cohomology [S23, §1.2, Thm. 1.6]:

$$\mathrm{H}^{2i-1}_{i-2,\text{nr}}(X^{\text{an}}_{\infty,s_\infty}, \mathbb{Q}/\mathbb{Z}(i))_{\text{ndiv}} \stackrel{def}{=} \mathrm{coker}[\mathrm{H}^{2i-1}_{i-2,\text{nr}}(X^{\text{an}}_{\infty,s_\infty}, \mathbb{Q}/\mathbb{Z}(i))_{\text{div}} \to \mathrm{H}^{2i-1}_{i-2,\text{nr}}(X^{\text{an}}_{\infty,s_\infty}, \mathbb{Q}/\mathbb{Z}(i))].$$

3.2. **Crystalline cohomology.** We now turn to the setting and retain the notation and conventions of Subsection 1.2.2.

3.2.1. *"Comparison" with crystalline cohomology.* A delicate issue when $p > 0$ is to find a suitable analogue of Artin's comparison isomorphism. Following the strategy of [A23], this will be achieved by combining Fact 12 below, which relies - *via* a L-function argument - on the Katz-Messing theorem [KM74] and comparison of various categories of isocrystals, with[7] the conjectural statement $\mathrm{CrysEt}_{\mathbb{Q}_\ell}(f, i)$.

[7]Note that [A23] was focussed on divisors, for which the fact that homological and numerical equivalence coincide is known.

Let $\mathscr{S}$ be a smooth, geometrically connected variety over F and consider a Cartesian square

$$\begin{tikzcd} \mathcal{X}_{\mathscr{S}} \arrow[r] \arrow[d, "f_{\mathscr{S}}"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ \mathscr{S} \arrow[r] & \mathcal{S}. \end{tikzcd}$$

Fact 12. [A23, Proof of Thm. 1.6.3.1 - esp. (2.1.2.1), Rem. 1.6.3.2] *Assume the canonical restriction morphism in étale $\mathbb{Q}_\ell$-cohomology*

$$\mathrm{H}^0(\mathcal{S}_{\bar{F}}, R^{2i}f_*\mathbb{Q}_\ell(i)) \tilde{\to} \mathrm{H}^0(\mathscr{S}_{\bar{F}}, R^{2i}f_*\mathbb{Q}_\ell(i))$$

is an isomorphism. Then the canonical restriction morphism in crystalline cohomology

$$\mathrm{H}^0(\mathcal{S}, R^{2i}f_{\mathrm{crys},*}\mathcal{O}_{\mathcal{X}/K}) \tilde{\to} \mathrm{H}^0(\mathscr{S}, R^{2i}f_{\mathscr{S},\mathrm{crys},*}\mathcal{O}_{\mathcal{X}_{\mathscr{S}}/K})$$

is an isomorphism.

3.2.2. *Proof of Lemma B (ii).* Let $s \in |S|^{\mathrm{gen}}_{\mathcal{V}_{\ell,\mathbb{Q}_\ell}}$. Recall we are to prove $V^a_{\mathbb{Q}_\ell,s} = \Lambda_{\mathbb{Q}_\ell}$. Replacing k, F by finite field extensions, one may assume there exists a smooth, separated and geometrically connected scheme $\mathscr{S}$ over F with generic point $\eta_{\mathscr{S}} : \mathrm{Spec}(k(s)) \to \mathscr{S}$ and such that $\mathscr{S}(F) \neq \emptyset$, and a Cartesian diagram

(10)
$$\begin{tikzcd} \mathcal{X}_t \arrow[r] \arrow[d, "f_t"'] \arrow[dr, phantom, "\square"] & \mathcal{X}_{\mathscr{S}} \arrow[r] \arrow[d, "f_{\mathscr{S}}"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X \arrow[l] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & X_s \arrow[l] \arrow[d, "f_s"] \arrow[lll, bend right] \\ F \arrow[r, "t"] \arrow[dr] & \mathscr{S} \arrow[r] \arrow[d] \arrow[dr] & \mathcal{S} \arrow[d] & S \arrow[l] \arrow[d] & k(s) \arrow[l, "s"'] \arrow[dl] \arrow[lll, bend left, "\eta_{\mathscr{S}}"] \\ & F & \mathscr{K} \arrow[l] & k \arrow[l, "\eta_{\mathscr{K}}"'] & \end{tikzcd}$$

Replacing further k, F by finite field extensions, one may assume that

$$V^a_{\mathbb{Q}_\ell,s} = \mathrm{im}[\mathrm{CH}^i(X_s) \to \mathrm{CH}^i(X_{\bar{s}}) \stackrel{c_{\ell,s}}{\to} V_{\mathbb{Q}_\ell}]. \tag{11}$$

From (11), it is enough to show that for every $\tilde{\alpha}_s \in \mathrm{CH}^i(X_s)_{\mathbb{Q}}$ with image $\alpha_{\ell,s} := c_{\ell,s}(\tilde{\alpha}_s) \in V_{\mathbb{Q}_\ell}$, there exists $\tilde{\alpha} \in \mathrm{CH}^i(X)_{\mathbb{Q}}$ such that $c_{\ell,s}(\tilde{\alpha}|_{X_s}) = \alpha_{\ell,s}$. We retain the notation and conventions in Diagram (10). Up to shrinking $\mathscr{S}$, one may assume there exists $\tilde{\alpha}_{\mathscr{S}} \in \mathrm{CH}^i(\mathcal{X}_{\mathscr{S}})_{\mathbb{Q}}$ such that $\tilde{\alpha}_{\mathscr{S}}|_{X_s} = \tilde{\alpha}_s$; write $\tilde{\alpha}_t := \tilde{\alpha}_{\mathscr{S}}|_{\mathcal{X}_t} \in \mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}}$. Consider now the canonical commutative diagram

$$\begin{tikzcd} \mathrm{CH}^i(\mathcal{X})_{\mathbb{Q}} \arrow[rrrr, "|_{\mathcal{X}_{\mathscr{S}}}"] \arrow[d, "c_{\mathrm{crys}}"'] \arrow[drr, "|_{\mathcal{X}_t}"] & & & & \mathrm{CH}^i(\mathcal{X}_{\mathscr{S}})_{\mathbb{Q}} \arrow[d, "c_{\mathrm{crys},\mathscr{S}}"] \arrow[dll, "|_{\mathcal{X}_t}"'] \\ \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}) \arrow[d, "\epsilon"'] \arrow[drr, "|_{\mathcal{X}_t}"] & & \mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} \arrow[d, "c_{\mathrm{crys},t}"'] & & \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}_{\mathscr{S}}) \arrow[d, "\epsilon"] \arrow[dll, "|_{\mathcal{X}_t}"'] \\ \mathrm{H}^0(\mathcal{S}, R^{2i}f_{\mathrm{crys},*}\mathcal{O}_{\mathcal{X}/K}) \arrow[rr] \arrow[rrrr, bend right, "\simeq"'] & & \mathrm{H}^{2i}_{\mathrm{crys}}(\mathcal{X}_t) & & \mathrm{H}^0(\mathscr{S}, R^{2i}f_{\mathscr{S},\mathrm{crys},*}\mathcal{O}_{\mathcal{X}_{\mathscr{S}}/K}). \arrow[ll] \end{tikzcd}$$

As $s \in S^{\mathrm{gen}}_{\mathcal{V}_{\ell,\mathbb{Q}_\ell}}$, the canonical restriction morphism

$$\mathrm{H}^0(\mathcal{S}_{\bar{F}}, R^{2i}f_*\mathbb{Q}_\ell(i)) \tilde{\to} \mathrm{H}^0(\mathscr{S}_{\bar{F}}, R^{2i}f_*\mathbb{Q}_\ell(i))$$

is an isomorphism - see [A23, §2.2.2]. Here, we implicity use the reduction 2.2.3 a), b). Hence, by Fact 12, the bottom horizontal arrow is an isomorphism. This implies that $\alpha_t := c_{\mathrm{crys},t}(\tilde{\alpha}_t)$ lies in $\mathrm{H}^0(\mathcal{S}, R^{2i}f_{\mathrm{crys},*}\mathcal{O}_{\mathcal{X}/K})$. But then, by implication 2) $\Longrightarrow$ 1) in $\mathrm{VCrys}(f,i)$, there exists $\tilde{\alpha}_{\mathcal{X}} \in \mathrm{CH}^i(\mathcal{X})_{\mathbb{Q}}$ such that $c_{\mathrm{crys},t}(\tilde{\alpha}_{\mathcal{X}}|_{\mathcal{X}_t}) =$

$c_{\mathrm{crys}}(\widetilde{\alpha}_{\mathcal{X}})|_{\mathcal{X}_t} = \alpha_t = c_{\mathrm{crys},t}(\widetilde{\alpha}_t)$. By $\mathrm{CrysEt}_{\mathbb{Q}_\ell}(f,i)$, this implies $c_{\ell,t}(\widetilde{\alpha}_{\mathcal{X}}|_{\mathcal{X}_t}) = c_{\ell,t}(\widetilde{\alpha}_t)$. The assertion thus follows, with $\widetilde{\alpha} = \widetilde{\alpha}_{\mathcal{X}}|_X$, from the canonical commutative specialization diagram of cycle class maps

$$\begin{tikzcd}
 & \mathrm{CH}^i(\mathcal{X})_{\mathbb{Q}} \arrow[dl, "|_{\mathcal{X}_t}"'] \arrow[d, "|_{X_s}"] \arrow[dr, "|_X"] & \\
\mathrm{CH}^i(\mathcal{X}_t)_{\mathbb{Q}} \arrow[d, "c_{\ell,t}"'] & \mathrm{CH}^i(X_s)_{\mathbb{Q}} \arrow[l, "sp_{s,t}"'] \arrow[d, "c_{\ell,s}"] & \mathrm{CH}^i(X)_{\mathbb{Q}} \arrow[l, "|_{X_s}"'] \\
\mathrm{H}^{2i}(\mathcal{X}_{\bar{t}}, \mathbb{Q}_\ell(i)) \arrow[r, equal] & \mathrm{H}^{2i}(X_{\bar{s}}, \mathbb{Q}_\ell(i)). &
\end{tikzcd}$$

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