Homework 4. Due by April, 11.

1. Prove that $\sum_{p\leq N} log^2 p = Nlog\, N + o(Nlog\, N)$.

2. For $\alpha \in \mathbb{R}$, let $||\alpha|| = inf(\{\alpha\}, 1-\{\alpha\})$. Prove that for $\alpha, \beta \in \mathbb{R}$, we have
$$||\alpha+\beta|| \leq ||\alpha|| + ||\beta||.$$

3. For a function f define $\Delta_d(f)(x) = f(x+d) - f(x)$. For $\ell \geq 2$, define
$$\Delta_{d_\ell, d_{\ell-1}, \dots d_1} = \Delta_{d_\ell} \circ \Delta_{d_\ell - 1} \circ \dots \Delta_1.$$
For $\ell \geq 1$ define $\Delta_\ell = \Delta_{1,\dots 1}$ (the composition of Δ_1 ℓ times). Prove that
   (a) $\Delta_\ell(f)(x) = \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} f(x+j)$.
   (b) Prove that
$$\Delta_{d_\ell, d_{\ell-1}, \dots d_1}(x^k) = \sum_{j_1+\dots+j_\ell+j=k,\ j\geq 0, j_1, \dots j_\ell \geq 1} \frac{k!}{j! j_1! \dots j_\ell!} d_1^{j_1} \dots d_\ell^{j_\ell} x^j.$$
   (Hint: use induction.)