This course provides an introduction to the Number Theory, with mostly analytic techniques. Topics include: primes in arithmetic progressions, zeta-function, prime number theorem, number fields, rings of integers, Dedekind zeta-function, introduction to analytic techniques: circle method, sieves.

Here is the some references (to be completed and extended):

1. S. J. Miller and R. Takloo-Bighash, *An Invitation to Modern Number Theory.*
2. A.Karatsuba, *Basic Analytic Number Theory.*

LECTURE 1

1 Primes in arithmetic progressions

The goal of this section is to prove the Dirichlet theorem:

Theorem 1.1 (Dirichlet). *Every arithmetic progression*

$$a, a+q, a+2q, \dots$$

in which a and q have no common factor, includes infinitely many primes.

1.1 Euler's identity and existence of infinitely many primes

The series $\sum_{n\geq 1} n^{-s}$ converges uniformly for s in a compact in the half-plane $Re\, s > 1$, so that it defines an analytic function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$$

(introduced by Riemann in 1859.)

Proposition 1.2. *The infinite product*

$$\prod_{p\ prime} (1-p^{-s})^{-1}$$

converges uniformly on any compact in the half-plane $Re\, s > 1$ and defines an analytic function verifying

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}.$$

Proof. We express $\frac{1}{1-p^{-s}}$ as a sum of a geometric series

$$\frac{1}{1-p^{-s}} = \sum_{m\geq 0} \frac{1}{p^{ms}}.$$

Let X be a sufficiently big integer. Multiplying the identities above for primes $\leq X$ we obtain:

$$\prod_{p\leq X}\frac{1}{1-p^{-s}}=\prod_{p\leq X}\sum_{m\geq 0}\frac{1}{p^{ms}}=\sum_{n\in N(X)}\frac{1}{n^s},$$

where $N(X)$ is the set of positive integers having all prime factors $\leq X$. Then for $Re\, s=t>1$ we have

$$|\zeta(s)-\prod_{p\leq X}\frac{1}{1-p^{-s}}|\leq\sum_{n\notin N(X)}\frac{1}{n^s}\leq\sum_{n>X}\frac{1}{n^t}.$$

To verify that the Euler product converges in remains to show that it is nonzero. Let us show that $\zeta(s)\neq 0$ for $Re\, s>1$. We use the Talyor series expansion for the principal definition of the complex logarithm: $ln(1-p^{-s})=-\sum_{m\geq 1}\frac{p^{-ms}}{m}$, so that for $Re\, s>1$ we obtain

$$\zeta(s)=exp(\sum_p\sum_{m\geq 1}\frac{p^{-ms}}{m})$$

is nonzero.

$\square$

The expression above provides a method to show the infinity of prime numbers. Write

$$ln\,\zeta(s)=\sum_p\sum_{m=1}^{\infty}m^{-1}p^{-ms}. \qquad (1)$$

Since $\zeta(s)\to\infty$ as $s\to 1$ from the right, and since

$$\sum_p\sum_{m=2}^{\infty}m^{-1}p^{-ms}<\sum_p\sum_{m=2}^{\infty}p^{-m}=\sum_p\frac{1}{p(p-1)}<1.$$

it follows that $\sum_p p^{-s}\to\infty$ as $s\to 1$ from the right. This proves the existence of an infinity of primes and, moreover, that the series $\sum p^{-1}$ diverges.

The proof Dirichlet theorem is inspired by the same idea, but with more involved techniques. We first investigate some additional properties of the zeta function.

1.2 Zeta function

Proposition 1.3. *Assume $s>1$. Then $lim_{s\to 1}(s-1)\zeta(s)=1$.*

Proof. We have

$$(n+1)^{-s}<\int_n^{n+1}t^{-s}dt<n^{-s}.$$

Taking the sum from 1 to ∞, one obtains

$$\zeta(s)-1<\int_1^{\infty}t^{-s}dt=\frac{1}{s-1}<\zeta(s).$$

Hence $1 < (s-1)\zeta(s) < s$. We ontain the result taking limit as $s \to 1$. $\square$

Corollary 1.4.
$$\frac{ln\zeta(s)}{ln(s-1)^{-1}} \overset{s\to 1}{\to} 1.$$

Proof. Denote $r(s) = (s-1)\zeta(s)$. Then $ln(s-1) + ln\zeta(s) = ln\, r(s)$, so that
$$\frac{ln\zeta(s)}{ln(s-1)^{-1}} = 1 + \frac{ln\, r(s)}{ln(s-1)^{-1}}.$$
By the proposition above, $r(s) \to 1$ as $s \to 1$. Hence $ln\, r(s) \to 0$ and we deduce the result. $\square$

Proposition 1.5.
$$ln\zeta(s) = \sum_p p^{-s} + R(s)$$
where $R(s)$ is bounded as $s \to 1$.

Proof. By proposition 1.2, we have $\zeta(s) = \prod_{p\le N}(1-p^{-1})^{-1}a_N(s)$, with $a_N(s) \to 1, N \to \infty$.

We then have
$$ln\zeta(s) = \sum_{p\le N}\sum_{m=1}^{N} m^{-1}p^{-ms} + ln\, a_N(s)$$
and, taking the limit for $N \to \infty$,
$$ln\zeta(s) = \sum_p p^{-s} + \sum_p \sum_{m=2}^{\infty} m^{-1}p^{-ms},$$
where the second sum is less than $\sum_p \sum_{m=2}^{\infty} p^{-ms} = \sum_p p^{-2s}(1-p^{-s})^{-1} \le (1-2^{-s})^{-1}\sum_p p^{-2s} \le 2\zeta(2)$. $\square$

If $s \in \mathbb{C}$, from the definition we see that $\zeta(s)$ is convergent for $Re\, s > 1$.

Proposition 1.6. *The function $\zeta(s) - (s-1)^{-1}$ can be continued to an analytic function on $\{s \in \mathbb{C}, Res > 0\}$.*

Proof. Assume $Re\, s > 1$. Then, using the lemma below, one can write
$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} n(n^{-s} - (n+1)^{-s}) = s\sum_{n=1}^{\infty} n\int_n^{n+1} x^{-s-1}dx =$$
$$= s\sum_{n=1}^{\infty}\int_n^{n+1}[x]x^{-s-1}dx = s\int_1^{\infty}[x]x^{-s-1}dx = s\int_1^{\infty}x^{-s}dx - s\int_1^{\infty}\{x\}x^{-s-1}dx =$$

$= \frac{s}{s-1} - s\int_1^\infty \{x\}x^{-s-1}dx$, where $[x]$ is the integral part of a real number x and $\{x\} = x - [x]$ is its fractional part. Since $0 \leq \{x\} \leq 1$ the last integral converges and defines an analytic function for $Re\, s > 0$ and the result follows. We obtain

$$\zeta(s) - \frac{s}{s-1} = 1 - s\int_1^\infty \{x\}x^{-s-1}dx.$$

□

Lemma 1.7. *Let $(a_n), (b_n)$ be two sequences of complex numbers such that $\sum a_n b_n$ converges. Let $A_n = \sum_1^n a_i$ and suppose $A_n b_n \to 0, n \to \infty$. Then*

$$\sum_{n=1}^\infty a_n b_n = \sum_{n=1}^\infty A_n(b_n - b_{n+1}).$$

Proof. Let $S_N = \sum_{n=1}^N a_n b_n$ and $A_0 = 0$. Then

$$S_N = \sum_{n=1}^N (A_n - A_{n-1})b_n = \sum_{n=1}^N A_n b_n - \sum_{n=1}^N A_{n-1}b_n = \sum_{n=1}^N A_n b_n - \sum_{n=1}^{N-1} A_n b_{n+1} = A_N b_N + \sum_{n=1}^{N-1} A_n(b_n - b_{n+1}).$$

The result follows taking the limit as $N \to \infty$. □

The following formula will be useful:

Corollary 1.8. *For $Re\, s > 0, N \geq 1$*

$$\zeta(s) = \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{1-s} - \frac{1}{2}N^{-s} + s\int_N^\infty \rho(x)x^{-s-1}dx,$$

with $\rho(x) = \frac{1}{2} - \{x\}$.

Proof. Write

$$\zeta(s) - (\sum_{n=1}^N \frac{1}{n^s} + \frac{N^{1-s}}{1-s} - \frac{1}{2}N^{-s} + s\int_N^\infty \rho(x)x^{-s-1}dx) =$$

$$= \frac{s}{s-1} - \sum_{n=1}^N \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + \frac{1}{2}N^{-s} - s\int_1^N (x - [x])x^{-s-1}dx + \int_N^\infty \frac{(-s)x^{-s-1}}{2}dx =$$

$$= \frac{s}{s-1} - \sum_{n=1}^N \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + \frac{1}{2}N^{-s} - \int_1^N sx^{-s}dx + \sum_{n=1}^{N-1} n(\frac{1}{n^s} - \frac{1}{(n+1)^s}) - \frac{1}{2}N^s =$$

$$= \frac{s}{s-1} - \sum_{n=1}^N \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + \frac{sN^{-s+1}}{s-1} - \frac{s}{s-1} + \sum_{n=1}^{N-1} \frac{1}{n^s} - \frac{N-1}{N^s} = 0.$$

□

1.3 Statement of Dirichlet theorem

Definition 1.9. Let S and T be two sets of positive integers, with T infinite. The **upper natural density** and **lower natural density** of S in T are defined as

$$\text{limsup}_{N\to\infty}\frac{\#\{n\in S, n\le N\}}{\#\{n\in T, n\le N\}} \text{ and } \text{liminf}_{N\to\infty}\frac{\#\{n\in S, n\le N\}}{\#\{n\in T, n\le N\}}.$$

If the upper and lower densities coincide, the common value is called **the natural density** of S in T.

Definition 1.10. Let S and T be two sets of positive integers, with $\sum_{n\in T} n^{-1}$ divergent. The **upper Dirichlet density** and **lower Dirichlet density** of S in T are defined as

$$\text{limsup}_{s\to+1}\frac{\sum_{n\in S} n^{-s}}{\sum_{n\in T} n^{-s}} \text{ and } \text{liminf}_{s\to+1}\frac{\sum_{n\in S} n^{-s}}{\sum_{n\in T} n^{-s}}.$$

If the upper and lower densities coincide, the common value is called **the Dirichlet density** $d(S)$ of S in T.

Note that Proposition 1.5 implies that a subset S of the set of all primes $\mathcal{P}$ has Dirichlet density if

$$\lim_{s\to 1}\frac{\sum_{p\in S} p^{-s}}{ln(s-1)^{-1}}$$

exists.

The following properties are straightforward:

Proposition 1.11. *Let $S\subset\mathcal{P}$.*

(i) If S is finite, then $d(S)=0$;

(ii) If S consists of all but finitely many primes, then $d(S)=1$;

(iii) If $S=S_1\cup S_2$, where S_1 and S_2 are disjoint and $d(S_1)$ and $d(S_2)$ both exist, then $d(S)=d(S_1)+d(S_2)$.

We will prove a more precise statement of the Dirichlet theorem:

Theorem 1.12 (Dirichlet)**.** *Let $a,q\in\mathbb{Z},(a,q)=1$. Let*

$$\mathcal{P}(a,q)=\{p \text{ prime }, p\equiv a \text{ mod } q\}.$$

Then $d(\mathcal{P}(a,q))=\frac{1}{\phi(q)}$, in particular, this set is infinite.

1.4 Characters

Let A be an abelian group.

Definition 1.13. A **character** on A is a group homomorphism $A \to \mathbb{C}^*$. The set of characters is denoted by $\hat{A}$.

Note that $\hat{A}$ is an abelian group: if $\chi, \psi \in \hat{A}$ we define $\chi\psi$ by $a \mapsto \chi(a)\psi(a)$. The trivial character χ_0, defined by $\chi_0(a) = 1$ for all $a \in A$, is the neutral element of the group. Finally, for $\chi \in \hat{A}$ we define χ^{-1} as the character given by $a \mapsto \chi(a)^{-1}$.

If A is a finite group of order n, we have $a^n = e$ for any $a \in A$ hence the values of χ are the roots of unity and $\overline{\chi(a)} = \chi(a)^{-1} = \chi^{-1}(a)$.

Proposition 1.14. *Let A be a finite abelian group. Then $A \simeq \hat{A}$.*

Proof. Suppose first that A is cyclic, generated by an element g of order n. Then any character χ is uniquely defined by its value $\chi(g)$. Since $\chi(g)$ is a root of unity, there are at most n characters. Now, if $\xi_n = e^{2\pi i/n}$ and λ is a character such that $\lambda(g) = \xi_n$, we obtain that the powers λ^k, $k = 1, \dots n$ are distinct characters, hence $\hat{A}$ is a cyclic group generated by λ. In the general case, since any finite abelian group is a direct product of cyclic groups, it is enough to check that if $A \simeq A_1 \times A_2$, then $\hat{A} \simeq \hat{A}_1 \times \hat{A}_2$, that we leave as an exercise.

□

Proposition 1.15. *Let A be a finite abelian group and $\chi, \psi \in \hat{A}$, $a, b \in A$. Then*

(i) $\sum_{a \in A} \chi(a)\overline{\psi(a)} = n\delta(\chi, \psi)$

(ii) $\sum_{\chi \in \hat{A}} \chi(a)\overline{\chi(b)} = n\delta(a, b)$.

Proof. (i) We have $\sum_{a \in A} \chi(a)\overline{\psi(a)} = \sum_a \chi\psi^{-1}(a)$. It is enough to show that $\sum_a \chi_0(a) = n$ and $\sum_a \chi(a) = 0$ if $\chi \neq \chi_0$. The first assertion follows from the definition of χ_0. For the second, we have that there is $b \in A$, $\chi(b) \neq \chi_0(b) = 1$. Then $\sum_a \chi(a) = \sum_a \chi(ba) = \chi(b) \sum_a \chi(a)$ and the result follows.

(ii The proof is similar to (i), using that if a is nonzero in A, there is a character ψ such that $\psi(a) \neq 0$. We leave it as an exercise.

□

Definition 1.16. A **Dirichlet character mod** m is a character for $A = (\mathbb{Z}/m\mathbb{Z})^*$ the group of units in the ring $\mathbb{Z}/m\mathbb{Z}$.

Note that Dirichlet characters mod m induce, and are induced from the characters $\chi : \mathbb{Z} \to \mathbb{C}^*$ such that

(i) $\chi(n + m) = \chi(n)$ for all $n \in \mathbb{Z}$;

(ii) $\chi(kn) = \chi(k)\chi(n)$ for all $k, n \in \mathbb{Z}$;

(iii) $\chi(n) \neq 0$ if and only if $(n, m) = 1$.

Since the order of the group $(\mathbb{Z}/m\mathbb{Z})^*$ is the value of Euler's function $\phi(m)$, there are $\phi(m)$ Dirichlet characters mod m. The proposition above gives in this case:

Proposition 1.17. *Let χ and ψ be Dirichlet characters modulo m and $a, b \in \mathbb{Z}$. Then*

(i) $\sum_{a=0}^{m-1} \chi(a)\overline{\psi(a)} = \phi(m)\delta(\chi, \psi)$

(ii) $\sum_{\chi} \chi(a)\overline{\chi(b)} = \phi(m)\delta(a, b)$.

1.5 L-functions

Let χ be a Dirichlet character modulo m.

Definition 1.18. The **Dirichlet L-function associated to χ** is

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

Note that since $|\chi(n)n^{-s}| \leq n^{-s}$, the function $L(s, \chi)$ converges and is continuous for $s > 1$.

Proposition 1.19. *(i)* $L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$*;*

(ii) $L(s, \chi_0) = \prod_{p|m} (1 - p^{-s})\zeta(s)$.

(iii) $\lim_{s\to 1}(s-1)L(s, \chi_0) = \phi(m)/m$. *In particular,* $L(s, \chi_0) \to \infty$ *as* $s \to 1$.

Proof. The statement (i) follows as in proposition 1.2. For(ii) write

$$L(s, \chi_0) = \prod_{(p,m)=1} (1 - \chi_0(p)p^{-s})^{-1} = \prod_{p|m}(1 - p^{-s}) \prod_{p}(1 - p^{-s})^{-1} = \prod_{p|m}(1 - p^{-s})\zeta(s)$$

using proposition 1.2 again. To establish (iii) we use proposition 1.3 and we obtain

$$\lim_{s\to 1}(s-1)L(s, \chi_0) = \prod_{p|m}(1 - p^{-1}) = \phi(m)/m.$$

□

Proposition 1.20. *Let χ be a nontrivial Dirichlet character modulo m. Then $L(s, \chi)$ can be continued to an analytic function for $Re\, s > 0$.*

Proof. Let $S(x) = \sum_{n \leq x} \chi(n)$. By lemma 1.7, we have

$$L(s, \chi) = \sum_{n=1}^{\infty} S(n)(n^{-s} - (n+1)^{-s}) = s \sum_{n=1}^{\infty} S(x) \int_{n}^{n+1} x^{-s-1} dx =$$

$= s \int_1^{\infty} S(x) x^{-s-1} dx$. By lemma below, $|S(x)| \leq \phi(m)$ for all x. Hence the above integral converges and defines an analytic function for $Re\, s > 0$. □

Lemma 1.21. *Let χ be a nontrivial character modulo m. For any $N > 0$*

$$|\sum_{n=0}^{N} \chi(n)| \leq \phi(m).$$

Proof. Let $N = qm + r, 0 \leq r < m$. Since $\chi(n+m) = \chi(n)$ and $\sum_{n=0}^{m-1} \chi(n) = 0$ by the orthogonality relations, we obtain

$$|\sum_{n=0}^{N} \chi(n)| = |q \sum_{n=0}^{m-1} \chi(n) + \sum_{n=0}^{r} \chi(n)| \leq |\sum_{n=0}^{r} \chi(n)| \leq \sum_{n=0}^{m-1} |\chi(n)| = \phi(m).$$

□

We now study Gauss sums associated to Dirichlet characters.

Definition 1.22. For χ a Dirichlet character we define $G(s, \chi) = \sum_p \sum_{k \geq 1} \frac{\chi(p^k) p^{-ks}}{k}$.

Note that since $|\frac{\chi(p^k) p^{-ks}}{k}| \leq p^{-ks}$ and $\zeta(s)$ converges for $s > 1$, the same holds for $G(s, \chi)$.

Proposition 1.23. *(i) For $s > 1$, $exp\, G(s, \chi) = L(s, \chi)$;*

(ii) $G(s, \chi) = \sum\limits_{(p,m)=1} \chi(p) p^{-s} + R_{\chi}(s)$, where $R_{\chi(s)}$ is bounded as $s \to 1$;

(iii)

$$\sum_{\chi} \overline{\chi(a)} G(s, \chi) = \phi(m) \sum_{p \equiv a(m)} p^{-s} + R_{\chi,a}(s), \tag{2}$$

where $R_{\chi,a}(s)$ is bounded as $s \to 1$.

(iv) $lim_{s \to 1} G(s, \chi_0)/ln(s-1)^{-1} = 1$.

Proof. Note that for $z \in \mathbb{C}, |z| < 1$ one has

$$exp(\sum_{k=1}^{\infty} \frac{z^k}{k}) = (1-z)^{-1}.$$

So that, for $z = \chi(p) p^{-s}$ we obtain $exp(\sum_{k=1}^{\infty} \frac{\chi(p^k) p^{-ks}}{k}) = (1 - \chi(p) p^{-1})^{-1}$ and we deduce (i).

The proof of (ii) is similar to propostion1.5. To get (iii), we multiply the both sides of (ii) by $\overline{\chi(a)}$ and sum over all Dirichlet characters modulo m:

$$\sum_{\chi} \overline{\chi(a)} G(s,\chi) = \sum_{(p,m)=1} p^{-s} \sum_{\chi} \overline{\chi(a)}\chi(p) + \sum_{\chi} \overline{\chi(a)} R_{\chi}(s).$$

By proposition 1.17, we obtain

$$\sum_{\chi} \overline{\chi(a)} G(s,\chi) = \phi(m) \sum_{p \equiv a(m)} p^{-s} + R_{\chi,a}(s),$$

where $R_{\chi,a}(s)$ is bounded as $s \to 1$, as required.

For (iv), we use that $L(s,\chi_0) = \prod_{p|m}(1-p^{-s})\zeta(s)$. Hence $G(s,\chi_0) = \sum_{p|m} ln(1-p^{-s}) + ln\zeta(s)$, so that the statement follows from Proposition 1.3. □

In particular, from (i) we obtain that the series $G(s,\chi)$ provides a definition for $lnL(s,\chi)$, with no choice of branch involved. Understanding the behaviour of $G(s,\chi)$ for χ non trivial is the crucial technical step in the proof of Dirichlet theorem. In lecture 3 we present a proof due to de la Vallée Poissin (1896).

Proposition 1.24. *Let $F(s) = \prod_{\chi} L(s,\chi)$ where the product is over all Dirichlet characters modulo m. Then, for s real and $s > 1$ we have $F(s) \geq 1$.*

Proof. By definition, $G(s,\chi) = \sum_p \sum_{k \geq 1} \frac{\chi(p^k)p^{-ks}}{k}$. Summing over χ and using Proposition 1.17, we obtain

$$\sum_{\chi} G(s,\chi) = \phi(m) \sum_{p^k \equiv 1(m)} \frac{1}{k} p^{-ks}.$$

The right-hand side of this equation is positive, taking the exponential, we obtain $\prod_{\chi} L(s,\chi) \geq 1$. □

Theorem 1.25. *Let χ be a nontrivial Dirichlet character modulo m. Then $L(1,\chi) \neq 0$.*

Proof. We first consider the case when χ is a complex character. By definition, for s real, we have $\overline{L(s,\chi)} = L(s,\bar{\chi})$. Letting $s \to 1$ we see that $L(1,\chi) = 0$ implies $L(1,\bar{\chi}) = 0$. Assume $L(1,\chi) = 0$. Since $L(s,\chi)$ and $L(s,\bar{\chi})$ have zero at $s = 1$, $L(a,\chi_0)$ has a simple pole at $s = 1$ by Proposition 1.19(iii) and the other factors are analytic at around $s = 1$ we obtain $F(1) = 0$. But from Proposition 1.24, for s real and $s > 1$ we have $F(s) \geq 1$, contradiction.

The case when χ is nontrivial real character (i.e. $\chi(n) = 0, 1$ or -1) is more difficult. For such character assume $L(1,\chi) = 0$ and consider

$$\psi(s) = \frac{L(s,\chi)L(s,\chi_0)}{L(2s,\chi_0)}.$$

Note that $\psi(s)$ is analytic for $Res > 1/2$: in fact, the zero of $L(s,\chi)$ at $s = 1$ cancels the simple pole of $L(s,\chi_0)$ and the denominator is analytic for $Re\, s > 1/2$. Moreover, since $L(2s,\chi_0)$ has a simple pole at $s = 1$ we have that $\psi(s) \to 0, s \to 1/2$.

Lemma 1.26. *For s real and $s > 1$ we have $\psi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ where $a_1 = 1$, $a_n \geq 0$ and the series is convergent for $s > 1$.*

Proof. We have

$$\psi(s) = \prod_p (1-\chi(p)p^{-s})^{-1}(1-\chi_0(p)p^{-s})^{-1}(1-\chi_0(p)p^{-2s}) = \prod_{p \nmid m} \frac{1-p^{-2s}}{(1-p^{-s})(1-\chi(p)p^{-s})}.$$

If $\chi(p) = -1$, the p-factor is equal to 1. Hence

$$\psi(s) = \prod_{\chi(p)=1} \frac{1+p^{-s}}{1-p^{-s}}.$$

We have $\frac{1+p^{-s}}{1-p^{-s}} = (1+p^{-s}) \sum\limits_{k=0}^{\infty} p^{-ks} = 1 + 2p^{-s} + 2p^{-2s} + \ldots$. Applying lemma 1.27 below yields the result. □

Expanding $\psi(s)$ (as a function of a complex variable) as a power series around $s = 2$, we obtain

$$\psi(s) = \sum_{m=0}^{\infty} b_m (s-2)^m.$$

Since $\phi(s)$ is analytic, the radius of convergence of this power series is at least 3/2. We have

$$b_m = \psi^{(m)}(2)/m! = \sum_{n=1}^{\infty} a_n(-ln\, n)^m n^{-2} = (-1)^m c_m, c_m \geq 0.$$

Hence $\phi(s) = \sum_{n=0}^{\infty} c_m(2-s)^m$ and $c_0 = \psi(2) = \sum_{n=1}^{\infty} a_n n^{-2} \geq a_1 = 1$. Hence for s real in $(\frac{1}{2}, 2)$ we have $\psi(s) \geq 1$, contradiction with $\psi(s) \to 0$ as $s \to 1/2$. This finishes the proof of the theorem. □

Lemma 1.27. *Let f be a nonnegative function on $\mathbb{Z}$ such that $f(mn) = f(m)f(n)$ for all $(m,n) = 1$. Assume that there is a constant c such that $f(p^k) < c$ for all prime powers p^k. Then*

(i) $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges for all real $s > 1$;

(ii) $\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p (1 + \sum_{k=1}^{\infty} f(p^k)p^{-ks})$.

Proof. Let $s > 1$ and $a(p) = \sum_{k=1}^{\infty} f(p^k)p^{-ks}$. Then

$$a(p) < cp^{-s} \sum_{k=0}^{\infty} p^{-ks} = cp^{-s}(1-p^{-s})^{-1},$$

so that $a(p) < 2cp^{-s}$. Since for $x > 0$ we have $1 + x < exp\, x$ we deduce

$$\prod_{p\le N}(1+a(p)) < \prod_{p\le N} expa(p) = exp\sum_{p\le N} a(p) < exp(2c\sum_p p^{-s}) := M.$$

By the definition of $a(p)$ and the multiplicativity of f we deduce

$$\sum_{n=1}^{\infty} f(n)n^{-s} < \prod_{p\le N}(1+a(p)) < M.$$

Since f is nonnegative, we obtain (i). We deduce (ii) similarly to Proposition 1.2. □

We now deduce as a corollary:

Proposition 1.28. *If χ is a nontrivial character modulo m, then $G(s,\chi)$ remains bounded as $s \to 1$ through real values $s > 1$.*

Proof. Since $L(1,\chi) \neq 0$ by theorem 1.25, there is a disk D around $L(1,\chi)$, not containing 0. Let $ln\, z$ be a single-valued branch of the logarithm, defined on D. Let $\delta > 0$ be such that $L(s,\chi) \in D$ for $s \in (1, 1+\delta)$. Then for s in this interval the exponential of both functions $lnL(s,\chi)$ and $G(s,\chi)$ is $L(s,\chi)$. Hence, there is an integer N such that for $s \in (1,1+\delta)$ one has

$$G(s,\chi) = 2\pi i N + lnL(s,\chi),$$

so that $\lim_{s\to 1} G(s,\chi)$ exists and is equal to $2\pi i N + lnL(1,\chi)$, in particular $G(s,\chi)$ is bounded. □

1.6 Proof of Dirichlet theorem

Recall the identity (2):

$$\sum_{\chi} \overline{\chi(a)} G(s,\chi) = \phi(m) \sum_{p\equiv a(m)} p^{-s} + R_{\chi,a}(s),$$

where $R_{\chi,a}(s)$ is bounded as $s \to 1$. We divide this identity by $ln(s-1)^{-1}$ and take the limit as $s \to 1$. By Proposition 1.28, the limit of the left-hand side is 1, and the limit of the right-hand-side is $\phi(m)d(\mathcal{P}(a,m))$. We obtain $d(\mathcal{P}(a,m)) = \frac{1}{\phi(m)}$ as claimed. □

2 Zeta function

Deeper properties concerning the distribution of primes are related to the properties of the zeta function. We continue investigating these properties using tools from real and complex analysis.

2.1 Fourier analysis

Definition 2.1. If $f \in L^1(\mathbb{R})$ we denote

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e^{-2\pi ixy}dx.$$

Examples:

- (Fourier inversion formula) If $f, \hat{f} \in L^1(\mathbb{R})$, then $f(x) = \int_{\mathbb{R}} \hat{f}(y)e^{2\pi ixy}dx$.
- for $f(x) = e^{-\pi x^2}$ one has $\hat{f}(y) = e^{-\pi y^2}$, i.e. one could think about this function as being 'self-dual'.

Let $\mathcal{L} \subset L^1(\mathbb{R})$ be the vector space of twice continuously differentiable functions, such that the functions f, f', f'' are rapidly decreasing (i.e. as $x^{-(1+\eta)}$ for some $\eta > 0$.)

Theorem 2.2. *(Poisson summation formula) For $f \in \mathcal{L}$, we have*

$$\sum_{m\in\mathbb{Z}} f(m) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

For the proof, see for example section 11.4.2 in [S. J. Miller and R. Takloo-Bighash, *An Invitation to Modern Number Theory*]. The formula holds under weaker assumptions, but the version above is enough for applications here.

Recall that the Γ-function is defined for $Re(s) > 0$ by

$$\Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt.$$

One has the following properties (see the next section for some of proofs):

- $\Gamma(n+1) = n!$ and $\Gamma(1) = 1$.
- $\Gamma(s)$ has a meromorphic continuation to the entire complex plane with simple poles at $s = 0, -1, -2, \ldots$ and the residue at $s = -k$ is $\frac{(-1)^k}{k!}$.
- (reflexion formula) $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$.
- (functional equation) $\Gamma(s+1) = s\Gamma(s)$.
- (duplication formula) $\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\pi^{\frac{1}{2}}\Gamma(2s)$, $s \in \mathbb{C} \setminus \mathbb{Z}$.
- (Stirling's formula) $log\Gamma(s) = (s-\frac{1}{2})log(s) - s + log\sqrt{2\pi} + \mathcal{O}(\frac{1}{|s|})$.
- $-\frac{\Gamma'(s)}{\Gamma(s)} = \frac{1}{s} + \gamma + \sum_{n=1}^{\infty}[\frac{1}{n+s} - \frac{1}{n}.]$
  As a consequence,
  $$\frac{\Gamma'(n)}{\Gamma(n)} = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k}.$$

Definition 2.3. For $Re(s) > 1$ define $\xi(s) = \frac{1}{2}s(s-1)\Gamma(\frac{s}{2})\pi^{-s/2}\zeta(s)$.

The following analytic continuation theorem is of high importance.

Theorem 2.4. *(Analytic continuation of the zeta function) The function $\xi(s)$ has an analytic continuation to an entire function and satisfies the functional equation*

$$\xi(s) = \xi(1-s).$$

Proof. By change of variables in the definition of the Gamma function we get

$$\int_0^\infty x^{\frac{1}{2}s-1} e^{-n^2\pi x} dx = \frac{\Gamma(\frac{s}{2})}{n^s \pi^{\frac{s}{2}}}.$$

Summing over $n \in \mathbb{N}$, for $Re(s) > 1$, we obtain

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1}(\sum_{n=1}^{\infty} e^{-n^2\pi x})dx = \int_0^\infty x^{\frac{1}{2}s-1} w(x)dx,$$

where $w(x) = \sum\limits_{n=1}^{\infty} e^{-n^2\pi x}$. Note that the absolute convergence of the sum justifies that one could exchange the order sum-integral in the first equality.

Diving the last integral into two pieces for $x > 1$ (resp. $x < 1$) and changing variables by $x \mapsto x^{-1}$ in the second we obtain:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \int_1^\infty x^{\frac{1}{2}s-1} w(x)dx + \int_1^\infty x^{-\frac{1}{2}s-1} w(\frac{1}{x})dx.$$

By lemma below, one deduces from the functional equation for $w(x)$ that

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}})w(x)dx.$$

Since $w(x)$ is rapidly decreasing, the integral on the right converges absolutely for any s and defines an entire function of s. The remaining assertions follow from the location of poles of $\frac{1}{s(s-1)}$ and the invariance of the right hand side of the last equality under the change $s \mapsto 1-s$. □

Lemma 2.5. *The function $w(x) = \sum\limits_{n=1}^{\infty} e^{-n^2\pi x}$ satisfies the functional equation*

$$w(\frac{1}{x}) = -\frac{1}{2} - \frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}} w(x).$$

Proof. Write $w(x) = \frac{\theta(x)-1}{2}$ with $\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x}$. Note that this series is converging rapidly for $x > 0$. By the Poisson summation formula, we have

$$\theta(x^{-1}) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x^{-1}} = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\pi t^2 x^{-1} + 2\pi i m t} dt = x^{\frac{1}{2}}\theta(x)$$

and the functional equation for $w(x)$ easily follows. □

Remark 2.6. Using the duplication and the reflexion formulas for the Γ-function, one could obtain the functional equation for the zeta function in the following form:

$$\zeta(s) = \frac{1}{\pi}(2\pi)^s \sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s).$$

Corollary 2.7. *$\zeta(-2m) = 0$ for all $m \in \mathbb{N}$.*

Proof. The result follows from the analytic continuation and the fact that the Γ-function has poles at $-m$, $m \in \mathbb{N}$. $\square$

The zeros $-2m$ of the zeta function given in the corollary above are called the **trivial zeros**. For $0 \leq Re(s) \leq 1$ the functional equation implies that zeros must lie symmetrically around the critical line $Re(s) = \frac{1}{2}$. The **Riemann Hypothesis** asserts that all zeros s of the zeta function with $0 \leq Re(s) \leq 1$ lie on the critical line.

Later we will establish:

Theorem 2.8. *(de la Vallée Poussin) There exists a constant $c > 0$ such that the zeta function has no zeros for*

$$Re(s) = \sigma \geq 1 - \frac{c}{log(|t|+2)}.$$

Corollary 2.9. *Let $T \geq 2$ and $c > 0$ a constant. Then for*

$$\sigma \geq 1 - \frac{c}{2\,log(T+2)},\ 2 \leq |t| \leq T$$

one has an estimation $|\frac{\zeta'(s)}{\zeta(s)}| = \mathcal{O}(log^2 T)$, where $s = \sigma + it$.

3 Distribution of primes

Let $\pi(x) = \sum\limits_{p \text{ prime },p\leq x} 1$. We will be interested in the asymptotic description of this function. First we need some facts on the Dirichlet sums.

Definition 3.1. The Dirichlet series is a series of the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{3}$$

where the coefficients a_n are complex numbers and $s = \sigma + it$.

To the Dirichlet series one associates the function

$$\Phi(x) = \sum_{n \leq x} a_n.$$

Theorem 3.2. *[Tauberian theorem] Assume that the series (3) converges for $\sigma > 1$, $|a_n| \leq A(n)$ where $A(n) > 0$ is a monotonic, increasing function and for $\sigma \to 1+0$ one has*

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} = \mathcal{O}((\sigma - 1)^{-\alpha}), \alpha > 0.$$

Then for any $b > 1$, $T \geq 1$, $x = N + \frac{1}{2}$ the following formula holds

$$\Phi(x) = \sum_{n \leq x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + \mathcal{O}(\frac{x^b}{T(b-1)^\alpha}) + \mathcal{O}(\frac{xA(2x)log(x)}{T}).$$

In addition, the constant in the $\mathcal{O}$-sign depends only on b_0.

Proof. First we prove that

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{a^s}{s} ds = \epsilon + \mathcal{O}(\frac{a^b}{T|log(a)|}) \tag{4}$$

where $\epsilon = 1$ if $a > 1$ and $\epsilon = 0$ if $0 < a < 1$. Let us consider the case $a > 1$ (we left the case $0 < a < 1$ as an exercise). Consider $U > b$ and the rectangular path Γ with sides $[-U + iT, -U - iT]$, $[-U - iT, b - iT]$, $[b - iT, b + iT]$, $[b + iT, -U + bT]$.

By Cauchy theorem, $\frac{1}{2\pi i} \int_\Gamma \frac{a^s ds}{s} = 1$, so that

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{a^s ds}{s} = 1 + R \tag{5}$$

where R is the opposite of the integral on the left, upper and bottom sides. The integrals on the upper and bottom sides have the same absolute value, so that on each of these sides we have

$$\frac{1}{2\pi} |\int \frac{a^s ds}{s}| \leq \frac{1}{2\pi} \int_{-U}^{b} \frac{a^\sigma d\sigma}{\sqrt{T^2 + \sigma^2}} \leq \frac{a^b}{Tlog(a)}.$$

Also we have for the left side

$$\frac{1}{2\pi} \int \frac{a^s ds}{s} \leq \frac{1}{2\pi} \int_{-T}^{+T} \frac{a^{-U} dt}{\sqrt{U^2 + t^2}} = \mathcal{O}(a^{-U}) \to 0$$

for $U \to \infty$. Passing to the limit in (5) when $U \to \infty$, we obtain the formula (4).

The series (3) is absolutely convergent for $s = b + it$. We obtain, exchanging the integral-sum:

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds = \sum_{n=1}^{\infty} a_n (\frac{1}{2\pi i} \int_{b-iT}^{b+iT} (\frac{x}{n})^s \frac{ds}{s}) = \sum_{n \leq x} a_n + R,$$

where

$$R = \mathcal{O}(\sum_{n=1}^{\infty} |a_n| (\frac{x}{n})^b T^{-1} |log \frac{x}{n}|^{-1}).$$

Note that, since $x = N + 1/2$, we have $x/n \neq 1$ for an integer n. We divide the sum under the $\mathcal{O}$-sign into two parts. For the first part, take $\frac{x}{n} \leq \frac{1}{2}$ or $\frac{x}{n} \geq 2$, so that $|log\frac{x}{n}| \geq 2$. From the assumptions $\sum_{n=1}^{\infty} \frac{|a_n|}{n^b} = \mathcal{O}(\frac{1}{(b-1)^\alpha})$, the first sum is $\mathcal{O}(\frac{x^b}{T(b-1)^\alpha})$. The remaining part is

$$\sum_{\frac{1}{2}x<n<2x} |a_n|(\frac{x}{n})^b T^{-1}|log\frac{x}{n}|^{-1}| \leq T^{-1}A(2x)2^b \sum_{\frac{1}{2}x<n<2x} |log\frac{N+0.5}{n}|^{-1}.$$

The summands with $n = N-1, N, N+1$ in the last sum are of order $\mathcal{O}(x)$ and for the remaining part r we obtain

$$r \leq \int_{x/2}^{N-1} (log\frac{N+0.5}{u})^{-1}du + \int_{N+1}^{2x} (log\frac{u}{N+0.5})^{-1}du = \mathcal{O}(x\,logx).$$

and the theorem follows. □

Now we are ready to prove the prime number theorem. Define $\Lambda(n) = log(p)$ if $n = p^k$ and $\Lambda(n) = 0$ otherwise.

Theorem 3.3. *There exists a constant $c > 0$ such that*

$$\psi(x) = \sum_{n \leq x} \Lambda(x) = x + \mathcal{O}(xe^{-c\sqrt{ln(x)}});$$

$$\pi(x) = \int_2^x \frac{du}{ln(u)} + \mathcal{O}(xe^{-\frac{c}{2}\sqrt{ln(x)}}).$$

Proof. For $Re(s) > 1$, using the Euler product argument (1) we write

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Using theorem 2.4, in the previous theorem we could take $\alpha = 1$, $A(n) = log(n)$. Consider $b = 1 + \frac{1}{log(x)}$,T= $e^{\sqrt{log(x)}}$. Then

$$\psi(x) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} (-\frac{\zeta'(s)}{\zeta(s)})\frac{x^s}{s}ds + \mathcal{O}(\frac{xln^2x}{T}).$$

By theorem 4.14 and its corollary, for some constant $c_1 > 0$, the zeta function has no zeros with $Re(s) = \sigma \geq \sigma_1 = 1 - \frac{c_1}{2log(T+2)}$, $|t| \leq T$, and $\frac{\zeta'(s)}{\zeta(s)} = \mathcal{O}(log^2T)$ for $s = \sigma \pm it$. Consider the integral

$$J = \frac{1}{2\pi i}\int_\Gamma (-\frac{\zeta'(s)}{\zeta(s)})\frac{x^s}{s}ds$$

along the rectangle Γ with sides $[\sigma_1 + iT, \sigma_1 - iT]$, $[\sigma_1 - iT, b - iT]$, $[b - iT, b + iT]$, $[b + iT;, \sigma_1 + iT]$.

Since the only nontrivial pole inside Γ of the function $(-\frac{\zeta'(s)}{\zeta(s)})\frac{x^s}{s}$ is $s=1$ with residue x, we have

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT}(-\frac{\zeta'(s)}{\zeta(s)})\frac{x^s}{s}ds = x+R$$

with R the sum of integrals along the left, upper and bottom sides. We will estimate these integrals. For the upper and bottom sides we have

$$|\frac{1}{2\pi\, i}\int_{\sigma_1+iT}^{b+iT}-\frac{\zeta'(s)}{\zeta(s)})\frac{x^s}{s}ds| \leq \int_{\sigma_1}^{b}|\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)})|\frac{x^\sigma}{T}d\sigma = \mathcal{O}(\frac{xlog^2T}{T}),$$

and the integral by the left side is

$$|\frac{1}{2\pi\, i}\int_{\sigma_1-iT}^{\sigma_1+iT}-\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds| = |\frac{1}{2\pi\, i}\int_{-T}^{T}\frac{\zeta'(\sigma_1+it)}{\zeta(\sigma_1+it)})\frac{x^{\sigma_i+it}}{\sigma_1+it}dt| =$$

$$= \mathcal{O}(x^{\sigma_1}log^2T(\int_0^1\frac{dt}{\sigma_1}+\int_1^T\frac{dt}{t})) = \mathcal{O}(x^{\sigma_1}log^3T).$$

From the inequalities above, the definition of T and σ_1 we deduce the first assertion of the theorem.

Consider

$$S = \sum_{n\leq x}\frac{\Lambda(n)}{log(n)} = \sum_{p\leq x}1 + \sum_{n=p^k,k\geq 2}\frac{\Lambda(n)}{log(n)}.$$

In the second sum $k \leq log(x)$ and for a fixed k we have at most $\sqrt{x}$ summands, ≤ 1. We deduce

$$S = \pi(x) + \mathcal{O}(\sqrt{x}log(x)). \tag{6}$$

In the lemma 3.4 below we put $c_n = \Lambda(n), f(x) = \frac{1}{log(x)}$, i.e. $C(x) = \sum_{n\leq x}c_n = \psi(x) = x + \mathcal{O}(xe^{-c\sqrt{log(x)}})$, $f'(x) = -\frac{1}{xlog^2x}$, so that we obtain

$$S = \int_2^x\frac{\psi(u)}{ulog^2u}du + \frac{\psi(x)}{log(x)} = \int_2^x\frac{du}{log^2u} + \frac{x}{log(x)} + R$$

with

$$R = \mathcal{O}(\int_2^x e^{-c\sqrt{log\,u}}\frac{du}{log^2u} + xe^{-c\sqrt{log\,x}}) =$$

$$= \mathcal{O}(\int_2^{\sqrt{x}}du + \int_{\sqrt{x}}^x e^{-c\sqrt{log\,u}}du + xe^{-c\sqrt{log\,x}}) = \mathcal{O}(xe^{-\frac{c}{2}\sqrt{ln(x)}})$$

and

$$\int_2^x\frac{du}{log^2u} + \frac{x}{log(x)} = -\frac{u}{log(u)}|_2^x + \int_2^x\frac{du}{log(u)} + \frac{x}{log(x)} = \int_2^x\frac{du}{log(u)} + \frac{2}{log\,2}.$$

The theorem follows from this equality and (6).

□

Lemma 3.4. *(Abel transform) Let $f(x)$ be a continuously differentiable function on the interval $[a, b]$, c_n be complex numbers and*

$$C(x) = \sum_{a<n\leq x} c_n.$$

Then

$$\sum_{a<n\leq b} c_n f(n) = -\int_a^b C(x)f'(x)dx + C(b)f(b).$$

Proof. We have

$$C(b)f(b) - \sum_{a<n\leq b} c_n f(n) = \sum_{a<n\leq b} c_n(f(b) - f(n)) =$$

$$= \sum_{a<n\leq b} \int_n^b c_n f'(x)dx = \sum_{a<n\leq b} \int_a^b c_n g(n, x)f'(x)dx,$$

where $g(n, x) = 1$ for $n \leq x \leq b$ and $g(n, x) = 0$ for $x < n$. To finish the proof of the lemma we exchange the order integral-sum in the last sum and notice that

$$\sum_{a<n\leq b} c_n g(n, x) = \sum_{a<n\leq x} c_n = C(x).$$

□

4 Zeros of the zeta-function

4.1 Entire functions

In this section we discuss properties of entire functions with prescribed set of zeros. More details could be found in [A.Karatsuba, *Basic Analytic Number Theory.*]

Theorem 4.1. *Let $a_1, \dots a_n, \dots$ be an infinite sequence of complex numbers with*

$$0 < |a_1| \leq |a_2| \leq \dots \leq |a_n| \leq \dots$$

and $\lim_{n\to\infty} \frac{1}{|a_n|} = 0$. Then there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ whose set of zeros coincide with set $\{a_n\}$ (with multiplicities).

Proof. For $n = 1, 2, \dots$ we set

$$u_n = u_n(s) = (1 - \frac{s}{a_n})exp(\frac{s}{a_n} + \frac{1}{2}(\frac{s}{a_n})^2 + \dots + \frac{1}{n-1}(\frac{s}{a_n})^{n-1}).$$

Consider the infinite product $\prod_{n=1}^{\infty} u_n(s)$. Let us show that the product converges for any $s \neq a_n$, and defines an entire function $g(s)$ with zeros $a_1, \dots a_n, \dots$. Consider a disk of radius $|a_n|$ and the product $\prod_{r=n}^{\infty} u_r(s)$. It is enough to establish that this product converges to an analytic function inside the disc $|s| < |a_n|$: in fact then the product $\prod_{n=1}^{\infty} u_n(s)$ is an analytic function in this disk, having only zeros a_i with $|a_i| < |a_n|$ and since $|a_n| \to \infty$, we deduce the theorem.

For $|s| < |a_n|$, $r \geq n$ we have

$$ln\, u_r(s) = ln(1 - \frac{s}{a_r}) + \frac{s}{a_r} + \frac{1}{2}(\frac{s}{a_r})^2 + \ldots + \frac{1}{r-1}(\frac{s}{a_r})^{r-1}.$$

Hence for $r = n, n+1, \ldots$ and $|s| < |a_n|$,

$$ln\, u_r(s) = -\frac{1}{r}(\frac{s}{a_r})^r - \frac{1}{r+1}(\frac{s}{a_r})^{r+1} - \ldots$$

and

$$u_r(s) = exp(-\frac{1}{r}(\frac{s}{a_r})^r - \frac{1}{r+1}(\frac{s}{a_r})^{r+1} - \ldots).$$

Hence it is enough to establish that the series

$$\sum_{r=n}^{\infty}[\frac{1}{r}(\frac{s}{a_r})^r + \frac{1}{r+1}(\frac{s}{a_r})^{r+1} + \ldots] \tag{7}$$

is absolutely convergent for $|s| < |a_n|$. But for any $0 < \epsilon < \frac{1}{2}$ and $|s| \leq (1-\epsilon)|a_n|$ we have

$$|\frac{1}{r}(\frac{s}{a_r})^r + \frac{1}{r+1}(\frac{s}{a_r})^{r+1} + \ldots| \leq \frac{1}{r}(1-\epsilon)^r + \frac{1}{r+1}(1-\epsilon)^{r+1} + \ldots < \frac{(1-\epsilon)^r}{\epsilon r}.$$

hence, using proposition 4.2 below, the series (7) is absolutely convergent for $|s| \leq (1-\epsilon)|a_n|$, so that we obtain that $\prod_{n=1}^{\infty} u_n(s)$ is analytic on $\mathbb{C}$ and we finish the proof of the theorem. $\square$

Proposition 4.2. *Let $u_n(s)$, $n \geq 1$ be an infinite sequence of analytic functions on the domain Ω, such that*

- $u_n(s) \neq -1$ *for all n and $s \in \Omega$;*
- $|u_n(s)| \leq a_n$ *for all n and $s \in \Omega$ and the series $\sum_{n=1}^{\infty} a_n$ converges.*

The the infinite product

$$\prod_{n=1}^{\infty}(1 + u_n(s))$$

converges for any $s \in \Omega$ and defines an analytic function $v(s)$ on Ω, such that $v(s) \neq 0$ for $s \in \Omega$.

Remark 4.3. If $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{1+s}} < \infty$, then the function

$$g(s) = \prod_{n=1}^{\infty} (1 - \frac{s}{a_n}) exp(\sum_{j=1}^{\infty} \frac{1}{j} (\frac{s}{a_n})^j)$$

satisfies the conditions of the theorem above.

One could also show that any entire function has the form

$$g(s) = e^{h(s)} s^m \prod_{n=1}^{\infty} (1 - \frac{s}{a_n}) exp(\sum_{j=1}^{\infty} \frac{1}{j} (\frac{s}{a_n})^j)$$

with h entire. This expression is more precise for functions of finite order.

Definition 4.4. Let $g(s)$ be an entire function and let $M(r) = M_g(r) = \max_{|s|=r} |g(s)|$. We say that g is an entire function of **finite order** if there exists $a > 0$ such that $M(r) < exp(r^a)$ for $r > r_0(a)$ for some constant $r_0(a)$. We then call $\alpha = \inf a$ the **order** of $g(s)$. If such a does not exists, we say that g is of infinite order.

Definition 4.5. Let $s_1, \dots s_n$ be a sequence of complex numbers, such that

$$0 < |s_1| \leq |s_2| \leq \dots \leq |s_n| \leq \dots .$$

If there exists $b > 0$ such that $\sum_{n=1}^{\infty} |s_n|^{-b} < \infty$ then we say that (s_n) has a finite order of convergence, and we call $\beta = inf b$ the **order** of convergence. If such b does not exist, we say that the order of convergence of (s_n) is ∞.
We have the following properties:

Theorem 4.6. *Let $g(s)$ be an entire function of finite order a, such that $g(0) \neq 0$ and let $s_1, \dots, s_n$ be zeros of g with $0 < |s_1| \leq |s_2| \leq \dots \leq |s_n| \leq \dots$. Then*

(i) the sequence s_n has a finite convergence order $\beta \leq \alpha$;

(ii) $g(s) = e^{h(s)} \prod_{n=1}^{\infty} (1 - \frac{s}{s_n}) exp(\sum_{j=1}^{p} \frac{1}{j} (\frac{s}{a_n})^j)$, where $p \geq 0$ is the smallest integer such that $\sum_{n=1}^{\infty} |s_n|^{-(p+1)} < \infty$ and $h(s)$ is a polynomial of degree $d \leq \alpha$ and $\alpha = \max(d, \beta)$.

(iii) If, in addition, for any $c > 0$ there is an infinite sequence $r_1, \dots r_n, \dots$ with $r_n \to \infty$ such that

$$\max |g(s)| > exp(c r_n^{\alpha}), |s| = r_n, n = 1, 2 \dots$$

then $\alpha = \beta$ and the series $\sum_{n=1}^{\infty} |s_n|^{-\beta}$ diverges.

4.2 Theorem of de la Vallée Poussin

Let $\xi(s)$ be defined as in theorem 2.4.

Theorem 4.7.
- *The function $\xi(s)$ is an entire function of order 1 with infinitely many zeros ρ_n such that $0 \leq Re\rho_n \leq 1$;*

- *the series* $\sum |\rho_n|^{-1}$ *diverges;*
- *the series* $\sum |\rho_n|^{-1-\epsilon}$ *converges for any* $\epsilon > 0$*;*
- *the zeros of* $\xi(s)$ *are nontrivial zeros of* $\zeta(s)$.

Proof. For $Re(s) > 1$ the zeta function, and, hence, the function $\xi(s)$ has no zeros. Theorem 2.4 implies that $\xi(s) \neq 0$ for $Re(s) > 0$ as well. Since $\xi(0) = \xi(1) \neq 0$, zeros of $\xi(s)$ coincide with nontrivial zeros of $\zeta(s)$.

To determine the order of $\xi(s)$, we consider $|s| \to \infty$. By corollary 1.8, $\zeta(s) = \mathcal{O}(|s|)$ for $Re(s) \geq \frac{1}{2}$. Since $|\Gamma(s)| \leq e^{c|s|\,|ln|s|}$, the order of ξ is at most one. But for $s \to +\infty$, $ln\Gamma(s) \equiv s\,ln(s)$, so that the order of $\xi(s)$ is 1. Theorem 4.6 imply that $\sum |\rho_n|^{-1}$, where ρ_n are zeros of $\xi(s)$ is divergent. In particular, $\xi(s)$ has infinitely many zeros, and the series $\sum |\rho_n|^{-1-\epsilon}$ is convergent for any $\epsilon > 0$. $\square$

Corollary 4.8. *(i)* $\xi(s) = e^{A+Bs} \prod_{n=1}^{\infty} (1 - \frac{s}{\rho_n}) e^{\frac{s}{\rho_n}}$*;*

(ii) nontrivial zeros of zeta-function are symmetric with respect to the lines $Re(s) = \frac{1}{2}$ *and* $Im(s) = 0$.

In what follows we enumerate zeros of zeta function in an increasing order (with respect to the absolute value).

Proposition 4.9.

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + \sum_{n=1}^{\infty} (\frac{1}{s-\rho_n} + \frac{1}{\rho_n}) + \sum_{n=1}^{\infty} (\frac{1}{s+2n} - \frac{1}{2n}) + B_0,$$

where ρ_n *are all nontrivial zeros of* $\zeta(s)$ *and* B_0 *is a constant.*

Proof. It is enough to take the logarithmic derivative in corollary 4.8(i). $\square$

Theorem 4.10. *Let* $\rho_n = \beta_n + i\gamma_n$*,* $n = 1, 2, \ldots$ *are all nontrivial zeros of* $\zeta(s)$*,* $T \geq 2$*. Then*

$$\sum_{n=1}^{\infty} \frac{1}{1 + (T - \gamma_n)^2} \leq c\, log\, T. \tag{8}$$

Proof. For $s = 2 + iT$, one has

$$|\sum_{n=1}^{\infty} (\frac{1}{s+2n} - \frac{1}{2n})| \leq \sum_{n \leq T} (\frac{1}{2n} + \frac{1}{2n}) + \sum_{n > T} \frac{|s|}{4n^2} \leq c_0\, log(T), \tag{9}$$

so that by proposition 4.9

$$-Re(\frac{\zeta'(s)}{\zeta(s)}) = Re(\frac{1}{s-1} - B_0 - \sum_{n=1}^{\infty}(\frac{1}{s+2n} - \frac{1}{2n}))-$$
$$- Re\sum_{n=1}^{\infty}(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}) \le c_1\, log(T) - Re\sum_{n=1}^{\infty}(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}).$$

From the Euler product expression (proposition 1.2), we have

$$-\frac{\zeta'(s)}{\zeta(s)}) = \sum_{n\ge 1}\frac{\Lambda(n)}{n^s}, \qquad (10)$$

where $\Lambda(n) = log(p), n = p^k$ and 0 otherwise. Hence

$$|\frac{\zeta'(s)}{\zeta(s)}| = |\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{2+iT}}| < c_2,$$

so that

$$Re\sum_{n=1}^{\infty}(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}) \le c_3\, log(T).$$

We deduce the theorem from the following inequalities

$$Re\frac{1}{s-\rho_n} = Re\frac{1}{(2-\beta_n)+i(T-\gamma_n)} = \frac{2-\beta_n}{(2-\beta_n)^2+(T-\gamma_n)^2} \ge$$
$$\ge \frac{0.5}{1+(T-\gamma_n)^2}$$

and $Re\frac{1}{\rho} = \frac{\beta_n}{\beta_n^2+\gamma_n^2} \ge 0$. □

Corollary 4.11. *The number of zeros ρ_n of the zeta function, such that*

$$T \le |Im(\rho_n)| \le T+1$$

is at most $c\, log(T)$.

Corollary 4.12. *For $T \ge 2$, one has* $\sum\limits_{|T-\gamma_n|>1} \frac{1}{|T-\gamma_n|^2} = \mathcal{O}(log(T))$.

Corollary 4.13. *For $-1 \le \sigma \le 2, s = \sigma + it, |t| \ge 2$, one has*

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{|t-\gamma_n|\le 1}\frac{1}{s-\rho_n} + \mathcal{O}(log|t|).$$

Proof. The inequality 9 is valid for $s = \sigma + it$, $|t| \geq 2$, $-1 \leq \sigma \leq 2$, so that

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}) + \mathcal{O}(log|t|).$$

We substract the same inequality for $s = 2 + it$:

$$\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}(\frac{1}{s-\rho_n} - \frac{1}{2+it-\rho_n}) + \mathcal{O}(log(|t|).$$

If $|\gamma_n - t| > 1$, then

$$|\frac{1}{\sigma + it - \rho_n} - \frac{1}{2+it-\rho_n}| \leq \frac{2-\sigma}{(\gamma_n - t)^2} \leq \frac{3}{(\gamma_n - t)^2}.$$

Now the statement follows from the previous corollaries 4.11 and 4.12. □

Theorem 4.14. *(de la Vallée Poussin) There exists a constant $c > 0$ such that the zeta function has no zeros for*

$$Re(s) = \sigma \geq 1 - \frac{c}{log(|t|+2)}.$$

Proof. The function $\zeta(s)$ has a pole at $s = 1$, hence for some γ_0 there is no zeros s with $|s-1| \leq \gamma_0$. Let $\rho_n = \beta_n + i\gamma_n$ be a zero of ζ with $|\gamma_n| > |\gamma_0|$. For $Re(s) = \sigma > 1$ we have as in (10)

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} = \sum_{n=1}^{\infty}\Lambda(n)n^{-\sigma}e^{-it\,log(n)},$$

so that

$$-Re\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\Lambda(n)n^{-\sigma}cos(tlog(n)).$$

Since for all real ϕ we have

$$3 + 4cos\,\phi + cos\,2\phi = 2(1 + cos\,\phi)^2 \geq 0,$$

we deduce

$$3(-\frac{\zeta'(\sigma)}{\zeta(\sigma)}) + 4Re(-\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}) + (-Re\frac{\zeta'(\sigma+i2t)}{\zeta(\sigma+i2t)}) \geq 0. \tag{11}$$

We will provide a majoration for each summand in the formula (11). By proposition 4.9 and corollary 4.11 for $s = \sigma$ and $1 < \sigma \leq 2$ we obtain

$$-\frac{\zeta'(s)}{\zeta(s)} < \frac{1}{\sigma - 1} + B_1,$$

where B_1 is a constant. For $s = \sigma + it, 1 < \sigma \leq 2, |t| > \gamma_0$ we find, again by proposition 4.9:

$$-Re\frac{\zeta'(s)}{\zeta(s)} < Alog(|t|+2) - \sum_{k=1}^{\infty} Re(\frac{1}{s-\rho_k} + \frac{1}{\rho_k}),$$

where $A > 0$ is an absolute constant. Since $0 \leq \beta_k \leq 1$, we have $\rho_k = \beta_k + i\gamma_k$ we deduce

$$Re\frac{1}{s-\rho_k} = Re\frac{1}{\sigma - \beta_k + i(t-\gamma_k)} = \frac{\sigma - \beta_k}{(\sigma-\beta_k)^2 + (t-\gamma_k)^2},$$

in addition $Re\frac{1}{\rho_k} = \frac{\beta_k}{\beta_k^2+\gamma_k^2} \geq 0$. We deduce

$$-Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} < A\,log(|t|+2) - \frac{\sigma-\beta_n}{(\sigma-\beta_n)^2+(t-\gamma_n)^2}$$

and

$$-Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} < A\,log(2|t|+2).$$

We now substitute these estimations in (11):

$$\frac{3}{\sigma-1} - 4\frac{\sigma-\beta_n}{(\sigma-\beta_n)^2+(t-\gamma_n)^2} + A_1\,log(|t|+2) \geq 0$$

for $A_1 > 0$ a constant. This inequality works for any t, $|t| > \gamma_0$ and any $\sigma, 1 < \sigma \leq 2$. For instance, for $t = \gamma_n, \sigma = 1 + \frac{1}{2A_1\,log(|\gamma_n|+2)}$, so that

$$\frac{4}{\sigma-\beta_n} \leq \frac{3}{\sigma-1} + A_1\,log(|\gamma_n|+2),$$

$$\beta_n \leq 1 - \frac{1}{14A_1\,log(|\gamma_n|+2)},$$

and we finish the proof of the theorem. □

Corollary 4.15. *Let $T \geq 2$ and $c > 0$ a constant. Then for*

$$\sigma \geq 1 - \frac{c}{2\,log(T+2)},\ 2 \leq |t| \leq T$$

one has an estimation $|\frac{\zeta'(s)}{\zeta(s)}| = \mathcal{O}(log^2 T)$, where $s = \sigma + it$.

Proof. Using corollary 4.13, we have

$$|\frac{\zeta'(s)}{\zeta(s)}| = \sum_{|t-\gamma_n|\leq 1} \frac{1}{s-\rho_n} + \mathcal{O}(log(T)).$$

Hence

$$|\frac{\zeta'(s)}{\zeta(s)}| \leq \sum_{|t-\gamma_n|\leq 1} \frac{1}{(\sigma-\beta_n)+i(t-\gamma_n)} + \mathcal{O}(log(T)).$$

Since $\beta_n \leq 1 - \frac{c}{log(T+2)}$ and $\sigma \geq 1 - \frac{c}{2log(T+2)}$, we have

$$|\frac{\zeta'(s)}{\zeta(s)}| \leq \frac{2}{c} log(T+2) \sum_{|t-\gamma_n| \leq 1} 1 + \mathcal{O}(log(T)) = \mathcal{O}(log^2 T).$$

. $\square$