HOMEWORK 9 MATH-GA 2350.001 DIFFERENTIAL GEOMETRY I

(due by December, 9, 2016)

1. Let M be a manifold of dimension $n \geq 2$ and x a point of M. Show that the inclusion $M \setminus \{x\} \to M$ induces an isomorphism in de Rham cohomology $H^k(M) \simeq H^k(M - \{x\})$, for any $k \neq n, n-1$.

2. (a) Let U be an open in $\mathbb{R}^n$. Show that $H^*(\mathbb{R} \times U)$ is isomorphic to $H^*(U)$.
   (b) Let $A \subset \mathbb{R}^n$ be a strict closed set (i.e. $A \neq \mathbb{R}^n$.) We embed A in $\mathbb{R}^{n+1}$ via the map $x \mapsto (x, 0)$. Show that
      i. $H^{p+1}(\mathbb{R}^{n+1} - A) \simeq H^p(\mathbb{R}^n - A)$ for $p \geq 1$;
      ii. $H^1(\mathbb{R}^{n+1} - A) \simeq H^0(\mathbb{R}^n - A)/\mathbb{R}$;
      iii. $H^0(\mathbb{R}^{n+1} - A) \simeq \mathbb{R}$.

3. Let M be a smooth manifold. Show that TM is orientable.

4. Let M, N be smooth compact oriented manifolds of dimension $n \geq 1$ with N connected and let $f : M \to N$ be a smooth map.
   (a) Show that there is a unique real number r such that the induced map $f^* : H^n(N) \to H^n(M)$ is a multiplication by r. We call $r = deg\, f$ the degree of f.
   (b) Show that the degree of a constant map is zero.
   (c) Show that if f is not surjective, then $deg\, f = 0$.
   (d) Show that degree is a homotopy invariant.
   (e) Show that any vector field on the sphere S^{2n} has at least one zero:
      i. Assume there is a nonwhere vanishing vector field X on S^{2n}. Show that we could assume that $X(x)$ is of norm 1, for any point $x \in S^{2n}$.
      ii. Consider the following map

      $$h : S^{2n} \times [0, 1] \to S^{2n}, (x, t) \mapsto (cos\pi t)x + (sin\pi t)X(x).$$

      Show that h is a homotopy between id and $-id$.
      iii. Conclude.