HOMEWORK 8 MATH-GA 2350.001 DIFFERENTIAL GEOMETRY I

(due by November, 28, 2016)

1. Let U be an open in $E = \mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a smooth function, $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i} \in \mathfrak{X}(U)$ be a vector field on U. Define: $grad\, f = \nabla f$ the vector field given by:
$$x \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$$
and $div\, X$ the function
$$x \mapsto \sum_{i=1}^n \frac{\partial X_i}{\partial x_i}.$$
Show that
   (a) $\mathcal{L}_X(dx_1 \wedge \ldots \wedge dx_n) = (div\, X) dx_1 \wedge \ldots \wedge dx_n$;
   (b) $div(fX) = f\, div X + \langle grad\, f, X \rangle$.

2. Let A be a ring. Consider the following commutative diagram, where the maps are the maps of A-modules:
$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[d] \\ B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & F' \end{tikzcd}$$
Show that it first, second, forth and fifth vertical maps are isomorphisms, then the third one is an isomorphism as well.