HOMEWORK 6 MATH-GA 2350.001 DIFFERENTIAL GEOMETRY I

(due by November, 14, 2016)

1. Let G be a Lie group. Show that there is a Lie group structure on the tangent bundle TG, such that the inclusion map $T_eG \to TG$ (where we viewT_eG as an additive Lie group on the vector space T_eG) and the projection $TG \to G$ are Lie group morphisms.

2. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. The goal of this exercise is to show that, for any $u \in \mathfrak{g}$, one has
$$T_u exp = T_e L_{exp\, u} \circ \theta(ad(-u)),$$
where $L_g : G \to G, x \mapsto gx$ is the left translation and $\theta : z \mapsto \frac{e^z-1}{z}$ is the series $\sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}$. Let $s, t \in \mathbb{R}$, $u, v \in \mathfrak{g}$.

   (a) Show that $exp(s+t)u = exp(su)exp(tu)$.

   (b) Denote, for $g, h \in G, w \in T_h G$
   $$g \cdot w = T_h L_g(w) \text{ and } w \cdot g = T_h R_{g^{-1}}(w).$$
   Show that
   $$(s+t)T_{(s+t)u} exp(v) = sT_{su} exp(v) \cdot exp(tu) + texp(su) \cdot T_{tu} exp(v).$$

   (c) Deduce that, if $f : \mathbb{R} \to End(\mathfrak{g})$ is the function
   $$f_u(s)(v) = s\, exp(-su) \cdot T_{su} exp(v),$$
   then
   $$f_u(s+t) = e^{ad(-tu)} \circ f_u(s) + f_u(t).$$

   (d) Deduce that $f_u'(s) = Id - ad(u) \circ f_u(s)$.

   (e) Check that the function $s \mapsto s\theta(s\, ad(-u))$ is also a solution of the differential equation in (d). Deduce that $f_u(s) = s\theta(s\, ad(-u))$.

   (f) Deduce that $T_u exp = T_e L_{exp\, u} \circ \theta(ad(-u))$.