HOMEWORK MATH-GA 2350.001 DIFFERENTIAL GEOMETRY I

(1, 2, 4 are due by September, 19, 2016, 3,5, 6 are due by September, 26, 2016)

1. Show that a topological manifold M is connected iff M is path-connected.

2. Let $\mathbb{R}P^n$ be the n-dimensional projective space, with the atlas given by the following functions

$$\phi_i : U_i \to \mathbb{R}^n, [x_1, \dots x_{n+1}] \mapsto (\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots \frac{x_{n+1}}{x_i}),$$

where U_i, $i = 1, \dots n+1$ are open subsets $\{[x_1, \dots, x_{n+1}], x_i \neq 0\}$.

   (a) Show that (U_i, ϕ_i) is a smooth atlas.

   (b) Show that $\mathbb{R}P^1$ is diffeomorphic to S^1.

   (c) Let $\pi : S^2 \to \mathbb{R}P^2$ be a map that sends a point (x, y, z) to a unique line through this point. Show that π is smooth and that π is local diffeomorphism: for any point $p \in S^2$ there exists an open neighborhood $U \subset M$ such that $\pi_U : U \to \pi(U)$ is a diffeomorphism on an open subset of $\mathbb{R}P^2$.

3. Let $0 < r \leq m \leq n$. Let $V_r \subset M_{n\times m}(\mathbb{R})$ be the set of matrices of rank r. Show that V_r is a smooth submanifold of $M_{n\times m}(\mathbb{R})$ and compute its dimension.

4. Let M be a manifold of class C^k. Let $A, B \subset M$ be closed subsets such that $A \cap B = \emptyset$. Show that there is a function $f \in C^k(M)$ with values in $[0, 1]$ and such that f is identically 0 on A and identically 1 on B.

5. Is product of two smooth manifolds with boundary a smooth manifold with boundary?

6. Let $n > 0$ be an integer and let $\langle , \rangle$ be the euclidean scalar product on $\mathbb{R}^{n+1}$, and $S^n := \{x \in \mathbb{R}^{n+1}, ||x|| = 1\}$. If $x \in \mathbb{R}^{n+1} \setminus 0$, we denote by $[x]$ the corresponding point of $\mathbb{RP}^n$.

   (a) Show that the map $f : S^n \times S^n \to \mathbb{R}$ defined by $(x, y) \mapsto \langle x, y\rangle$ is smooth. Find all points were it is a submersion.

   (b) Let $M \subset S^n \times S^n$ consists of orthogonal couples. Show that M is a smooth submanifold[1] of $S^n \times S^n$.

   (c) Let $M' \subset \mathbb{RP}^n \times \mathbb{RP}^n$ consists of couples of orthogonal lines (L, L') of $\mathbb{R}^{n+1}$. Show that M' is a smooth submanifold of $\mathbb{RP}^n \times \mathbb{RP}^n$.

   (d) Let E be the set of triples (x, x', y) of $S^n \times S^n \times \mathbb{R}^{n+1}$ such that $\langle x, x'\rangle = \langle x, y\rangle = \langle x', y\rangle = 0$ and $\pi : E \to M$ be the map $(x, x', y) \mapsto (x, x')$. Show that π is a smooth vector bundle over M.

[1] A subset $M \subset N$ of a smooth n-manifold N is a smooth submanifold if for any point $p \in M$ there is a chart (U, ϕ) of N at p such that $\phi(U \cap M)$ is a submanifold of $\phi(U) \subset \mathbb{R}^n$ in the sense of one of the four equivalent definitions given during the lecture

(e) Let E' be the set of couples $([x], y)$ of $\mathbb{RP}^n \times \mathbb{R}^{n+1}$ such that $\langle x, y \rangle = 0$ and $\pi' : E' \to \mathbb{RP}^n$ be the map $([x], y) \mapsto [x]$. Show that π' is a smooth vector bundle.

(f) Let E'' be the set of couples $(([x], y), ([x'], y'))$ of $E' \times E'$ such that $\langle x, x' \rangle = 0$ and $y = y'$ and let $\pi'' : E'' \to M$ be the map defined by $(([x], y), ([x'], y')) \mapsto ([x], [x'])$. Show that π'' is a is a smooth vector bundle.