HOMEWORK V
MATH-UA 0248-001 THEORY OF NUMBERS

due on October, 13, 2017

1. Solve each of the following sets of simultaneous congruences:
   (a) $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$;
   (b) $2x \equiv 1 \pmod 5$, $3x \equiv 9 \pmod 6$, $4x \equiv 1 \pmod 7$, $5x \equiv 9 \pmod{11}$.
2. For positive integers m and n prove that $\phi(m)\phi(n) = \phi(mn)\phi(d)/d$ where $d = gcd(m, n)$.
3. If m, n are relatively prime positive integers, prove that
$$m^{\phi(n)} + n^{\phi(m)} \equiv 1 \,(\text{mod } mn).$$
4. Let a be a natural number whose last digit is an element in the set $\{1, 3, 7, 9\}$. Prove that the last two digits of a^{41} are the same as those of a.
5. If p is an odd prime, prove that the congruence
$$x^{p-2} + \ldots + x^2 + x + 1 \equiv 0 \,(\text{mod } p)$$
has exactly $p - 2$ incongruent solutions and they are $2, 3, \ldots, p - 1$.
6. Consider the congruence $x^2 - 1 \equiv 0 \pmod 8$. How many solutions does it have with $0 \leq x < 8$?