HOMEWORK IV
MATH-UA 0248-001 THEORY OF NUMBERS
due on October, 6, 2017

1. Compute the following values of the Euler's Phi Function:
   (a) $\phi(125)$;
   (b) $\phi(120)$;
   (c) $\phi(10^k)$, where k is a positive integer.

2. If p and q are distinct odd primes, prove $2^{pq+1} \equiv 2^{p+q} \pmod{pq}$.

3. (a) Suppose that $p_1, p_2, \ldots p_r$ are distinct primes that divide n. Show that the following formula for $\phi(n)$ is correct.

$$\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_r}).$$

   (b) Find all values n that solve each of the following equations.
      i. $\phi(n) = \frac{n}{2}$,
      ii. $\phi(n) = \frac{n}{3}$,
      iii. $\phi(n) = \frac{n}{6}$.

4. If p is a prime, prove that

$$p \mid a^p + a \cdot (p-1)! \text{ and } p \mid a^p \cdot (p-1)! + a$$

   for any integer a.

5. Using Wilson's Theorem, prove that

$$1^2 \cdot 3^2 \cdot 5^2 \cdot (p-2)^2 \equiv (-1)^{\frac{(p+1)}{2}} (\text{mod } p)$$

   for $p > 2$ a prime number. (Hint: use that $k \equiv -(p-k) \pmod{p}$.)