Exercises for the preparation to the final exam

1. Solve the system of congruence equations:

$$\begin{cases} x \equiv 1 \ (\text{mod } 4) \\ x \equiv 5 \ (\text{mod } 7) \\ x \equiv 4 \ (\text{mod } 9) \end{cases}$$

2. Does the following congruence equation have a solution?

   (a) $13x \equiv 1(\text{mod } 26)$;

   (b) $13x \equiv 1(\text{mod } 29)$.

3. Find all integers $50 \leq a < 100$ such that a^{44} has remainder 4 when divided by 43.

4. (a) Is 7 a square modulo 23?

   (b) Is 7 a square modulo 209?

   (c) Does the congruence $x^2 - 2x + 1 \equiv 7(\text{mod } 23)$ have a solution?

5. Recall that $\sigma(n)$ denotes the sum of divisors of n. Find all pairs of twin primes p and $q = p + 2$ such that $2^{\sigma(pq)} \equiv 16(\text{mod } p)$. (Recall that p and q are called twin primes if $q = p + 2$)

6. Which of the following equations have integral solutions? How many solutions (finitely many or infinitely many)?

   (a) $x^2 - 10y^2 = 1$;

   (b) $x^2 - 9y^2 = 1$;

   (c) $4x^2 + y^2 = z^2$.