

MATH-GA2210.001 : Final Exam, May, 11, 2016

Lecture notes are authorized.

The exam is posted online on May, 11, 2016, by 7 pm.

The copies should be sent by mail (either a scan, or a pdf file) to
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The deadline to send the corrections is May, 12, 2016, 7pm.

1. (a) Recall that

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

Show that

$$\text{lcm}\{1, 2, \dots, n\} = e^{\psi(n)},$$

where the left hand side is the least common multiple of the integers
 $1, 2, \dots, n$.

(b) Show that

$$e^{\psi(2n+1)} \int_0^1 x^n (1-x)^n dx$$

is a positive integer. Deduce that

$$\psi(2n+1) \geq 2n \log 2.$$

2. (a) Let $K = \mathbb{Q}_p(\zeta_p)$, where ζ_p is a primitive p^{th} root of unity. Show that
 $[K : \mathbb{Q}_p] = p - 1$ and that this extension is totally ramified.

(b) (*) Which quadratic extensions of \mathbb{Q}_2 are ramified?

3. Let $K = \mathbb{Q}(\sqrt{-31})$. Let $\alpha = \frac{1+\sqrt{-31}}{2}$.

(a) Determine \mathcal{O}_K and $\text{disc}(\mathcal{O}_K)$.

(b) Determine the ideals of \mathcal{O}_K containing 2 or 3. Among these ideals, which
are principal (resp. prime)?

(c) Decompose the ideals (2), (3), and (α) as a product of prime ideals in
 \mathcal{O}_K .

(d) Determine the group $Cl(\mathcal{O}_K)$.