FINAL EXAM MATH-GA.2130-001 ALGEBRA I

December, 17, 2020, 7.10-9pm

CLASS NOTES AND BOOKS ARE AUTHORIZED.

1. Let G be an abelian group of order $\#G = n$ and let $H \subset G$ be a subgroup of order $\#H = m$. Assume that $\frac{n}{m} = d_1 d_2$ where $(d_1, d_2) = 1$. Show that the quotient group G/H is isomorphic to $C_1 \oplus C_2$ with $\#C_i = d_i$, $i = 1, 2$.

2. Let k be a field, let $A = k[y]$ be the ring of polynomials in one variable over k, and let $L = k(y)$ be its field of fractions. Let $X = V(x^3 - y) \in \mathbb{A}^2_k$, $k[X] = k[x, y]/(x^3 - y)$ and let $F = Frac(k[X])$ be the field of fraction of $k[X]$. For each of the following statements, determine if

   - the statement is always true (give proof)
   - the statement is always false (give a counterexample or proof)
   - the statement could be both true or false (give examples for both cases)

   (a) The ring $k[X]$ is factorial.

   (b) X is an irreducible variety.

   (c) $[F : L] = 3$.

   (d) F/L is a Galois extension. If so, determine the Galois group.

3. Let L/k be a Galois extension with the Galois group G of order 63. Show that there is a subextension $k \subset F \subset L$ such that L/F is Galois and $[L : F] = 7$. Is F/k Galois?

4. Let $P = x^5 - 5x^2 + 1$ and let G be the Galois group of P over $\mathbb{Q}$.

   (a) Show that P is separable and that there is an injective morphism of groups $G \to S_5$.

   (b) Show that the reduction $\bar{P}$ in $\mathbb{F}_2[x]$ has not roots in $\mathbb{F}_2$ and in $\mathbb{F}_4$. Deduce that $\bar{P}$ is irreducible.

   (c) Show that P is irreducible in $\mathbb{Z}[x]$; deduce that P is irreducible in $\mathbb{Q}[x]$.

   (d) Show that G has an element of order 5 using a mod p reduction argument for a suitable choice of p.

   (e) Let σ be the complex conjugation. Show that $\sigma \in G$.

   (f) Determine G. (you can admit that a transposition and a 5-cycle generate S_5).